

Regularity Analysis for an Abstract System of Coupled Hyperbolic and Parabolic Equations

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Abstract

In this paper, we provide a complete regularity analysis for the following abstract system of coupled hyperbolic and parabolic equations

$$\begin{cases} u_{tt} = -Au + \gamma A^\alpha w, \\ w_t = -\gamma A^\alpha u_t - kA^\beta w, \\ u(0) = u_0, \quad u_t(0) = v_0, \quad w(0) = w_0, \end{cases}$$

where A is a self-adjoint, positive definite operator on a complex Hilbert space H , and $(\alpha, \beta) \in [0, 1] \times [0, 1]$. We are able to decompose the unit square of the parameter (α, β) into three parts where the semigroup associated with the system is analytic, of specific order *Gevrey* classes, and non-smoothing, respectively. Moreover, we will show that the orders of *Gevrey* class are sharp, under proper conditions.

Keywords: hyperbolic-parabolic equations, analytic semigroup, Gevrey class semigroup

MSC (2010): 35B65, 35K90, 35L90, 47A10, 47D06, 93D20

1 Introduction

Let H be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. We consider the following abstract system of coupled hyperbolic and parabolic equations:

$$\begin{cases} u_{tt} = -Au + \gamma A^\alpha w, \\ w_t = -\gamma A^\alpha u_t - kA^\beta w, \\ u(0) = u_0, \quad u_t(0) = v_0, \quad w(0) = w_0, \end{cases} \quad (1.1)$$

where A is a self-adjoint, positive definite (unbounded) operator on a complex Hilbert space H ; $\gamma \neq 0$, $k > 0$, and $\alpha, \beta \in [0, 1]$ are fixed real numbers. Our main interest is the regularity of the solution to this system in terms of the parameters α, β .

We define

$$\mathcal{H} = \mathcal{D}(A^{\frac{1}{2}}) \times H \times H.$$

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Any element in \mathcal{H} is denoted by $U = (u, v, w)^T$. Introduce

$$\langle U_1, U_2 \rangle_{\mathcal{H}} = \langle A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}u_2 \rangle + \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle, \quad \forall U_i = \begin{pmatrix} u_i \\ v_i \\ w_i \end{pmatrix} \in \mathcal{H}, \quad i = 1, 2.$$

Then $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner product under which \mathcal{H} is a Hilbert space. By denoting $v = u_t$ and $U_0 = (u_0, v_0, w_0)^T$, system (1.1) can be written as an abstract linear evolution equation on the space \mathcal{H} ,

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}_{\alpha, \beta} U(t), & t \geq 0, \\ U(0) = U_0, \end{cases} \quad (1.2)$$

where the operator $\mathcal{A}_{\alpha, \beta} : \mathcal{D}(\mathcal{A}_{\alpha, \beta}) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ is defined by

$$\mathcal{A}_{\alpha, \beta} = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & \gamma A^{\alpha} \\ 0 & -\gamma A^{\alpha} & -k A^{\beta} \end{pmatrix}, \quad (1.3)$$

with the domain

$$\mathcal{D}(\mathcal{A}_{\alpha, \beta}) = \mathcal{D}(A) \times \mathcal{D}(A^{\alpha \vee \frac{1}{2}}) \times \mathcal{D}(A^{\alpha \vee \beta}), \quad (1.4)$$

where $a \vee b = \max\{a, b\}$ for any $a, b \in \mathbb{R}$. It is known that $\mathcal{A}_{\alpha, \beta}$ (which is identified with its closure) generates a C_0 -semigroup $e^{\mathcal{A}_{\alpha, \beta} t}$ of contractions on \mathcal{H} ([1]). Then the solution to the evolution equation (1.2) admits the following representation:

$$U(t) = e^{\mathcal{A}_{\alpha, \beta} t} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (1.2). With this in hand, regularity and stability are the most interesting properties for the solutions to evolution equations that attract people's attention. Before going further, let us recall some definitions relevant to the regularity and stability of C_0 -semigroups.

Definition 1.1. Let e^{At} be a C_0 -semigroup on a Hilbert space \mathcal{H} .

(i) Semigroup e^{At} is said to be *analytic* if there exists an extension $T(\tau)$ of e^{At} to the following set

$$\Sigma_{\theta} \equiv \{\tau \in \mathbb{C} \mid |\arg \tau| < \theta\} \cup \{0\},$$

for some $\theta \in (0, \frac{\pi}{2})$ so that for any $x \in \mathcal{H}$, $\tau \mapsto T(\tau)x$ is continuous on Σ_{θ} satisfying the following semigroup property

$$T(\tau_1 + \tau_2) = T(\tau_1)T(\tau_2), \quad \forall \tau_1, \tau_2 \in \Sigma_{\theta}, \text{ with } \tau_1 + \tau_2 \in \Sigma_{\theta},$$

and $\tau \mapsto T(\tau)$ is analytic over $\Sigma_{\theta} \setminus \{0\}$ in the uniform operator topology of $\mathcal{L}(\mathcal{H})$ (the space of all linear bounded operators from \mathcal{H} to \mathcal{H}).

(ii) Semigroup e^{At} is said to be of *Gevrey* class δ (with $\delta > 1$) if it is infinitely differentiable and for any compact set $\mathcal{K} \subset (0, \infty)$ and any $\theta > 0$, there exists a constant $K = K(\theta, \mathcal{K})$, such that

$$\|\mathcal{A}^n e^{At}\|_{\mathcal{L}(\mathcal{H})} \leq K \theta^n (n!)^{\delta}, \quad \forall t \in \mathcal{K}, \quad n \geq 0. \quad (1.5)$$

(iii) Semigroup e^{At} is said to be *differentiable* if for any $x \in \mathcal{H}$, $t \mapsto e^{At}x$ is differentiable on $(0, \infty)$.

(iv) Semigroup $e^{\mathcal{A}t}$ is said to be *exponentially stable* with decay rate $\omega > 0$ if there exists a constant $M \geq 1$ such that

$$\|e^{\mathcal{A}t}\| \leq Me^{-\omega t}, \quad t \geq 0.$$

(v) Semigroup $e^{\mathcal{A}t}$ is said to be *polynomially stable of order $j > 0$* if there exists a constant $M > 0$ such that

$$\|e^{\mathcal{A}t}\mathcal{A}^{-1}\| \leq Mt^{-j}, \quad t > 0.$$

In the above, the first three notions are about the regularity of C_0 -semigroups and the last two notions are about the asymptotically stability of C_0 -semigroups. We will see shortly that these notions are intrinsically related. Note that in (1.5), if $\delta = 1$, then the semigroup is analytic.

We now briefly recall some history. In 1981, Chen–Russell ([3]) considered the abstract elastic system with direct damping (the so-called linear oscillator) of following form:

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A}_\alpha \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & -B_\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (1.6)$$

on $\mathcal{H} = D(A^{\frac{1}{2}}) \times H$, where both A and B_α are (unbounded) positive definite on a Hilbert space H . Two conjectures for the analyticity of the associated C_0 -semigroup $e^{\mathcal{A}_\alpha t}$ were posed. It was shown by Huang [8, 9] and Huang–Liu [10] that if B_α is equivalent to A^α (in a certain sense) with $\frac{1}{2} \leq \alpha \leq 1$, the semigroup $e^{\mathcal{A}_\alpha t}$ is analytic and exponentially stable. Complete regularity results for such a system were obtained by Chen–Triggiani ([4, 5]), which says: When B_α is equivalent to A^α (in a certain sense), the associated C_0 -semigroup $e^{\mathcal{A}_\alpha t}$ is analytic for $\frac{1}{2} \leq \alpha \leq 1$, is of Gevrey class $\delta > \frac{1}{2\alpha}$ for $0 < \alpha < \frac{1}{2}$.

Having the complete results for system (1.6), people naturally turned the attention to thermoelastic equations, such as string, beam and plate, and so on. In the early 1990's, Russell [17] proposed an abstract system of a second order conservative equation coupled with a first order dissipative equation:

$$\frac{d}{dt} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathcal{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & B \\ 0 & -B^* & -D \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \quad (1.7)$$

This can be regarded as a system with indirect damping and velocity coupling. He pointed out that it is desirable to obtain complete results for system (1.7) similar to the known results for system (1.6). This has motivated studies of system (1.7) and/or (1.1) since then. For (1.1), a complete stability analysis was carried out by the first two authors of the current paper in 2013 (see [11]). To state the result, let us introduce the following sets which give a partition of the unit square $[0, 1] \times [0, 1]$:

$$\begin{cases} S = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid |2\alpha - 1| \leq \beta \leq 2\alpha \right\}, \\ S_1 = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid 2\alpha \vee \frac{1}{2} < \beta \right\}, \\ S_2 = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid \beta < 1 - 2\alpha, \beta \leq \frac{1}{2} \right\}, \\ S_3 = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid \beta < 2\alpha - 1 \right\}, \end{cases} \quad (1.8)$$

where $a \wedge b = \min\{a, b\}$, and we recall that $a \vee b = \max\{a, b\}$. See Figure 1. Note that

$$[0, \frac{1}{4}) \times \{\frac{1}{2}\} \subseteq S_2.$$

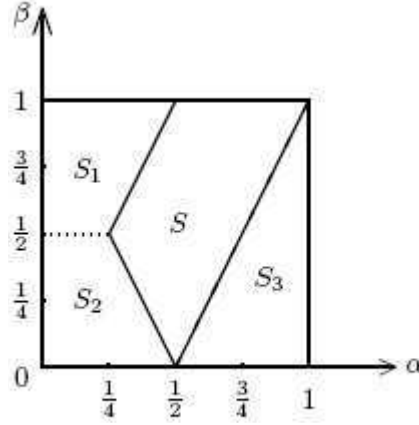


Figure 1: Region of stability

Here is the stability result found in [11].

Theorem 1.1. *The semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$ has the following stability properties:*

- (i) *In S , it is exponentially stable;*
- (ii) *In $S_1 \cup S_2$, it is polynomially stable of order $\frac{1}{2(\beta-2\alpha)} \wedge \frac{1}{2-2(2\alpha+\beta)}$;*
- (iii) *In S_3 , it is not asymptotically stable.*

Note that

$$\frac{1}{2(\beta-2\alpha)} \wedge \frac{1}{2-2(2\alpha+\beta)} = \begin{cases} \frac{1}{2(\beta-2\alpha)} > 0, & (\alpha, \beta) \in S_1, \\ \frac{1}{2-2(2\alpha+\beta)} > 0, & (\alpha, \beta) \in S_2. \end{cases}$$

For the regularity of the semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$, we recall the following results from the literature.

- In 1996, Muñoz Rivera and Racke studied the smoothing property of the semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$ ([15]). It was shown that this semigroup is C^∞ in the region

$$S^o = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid |1 - 2\alpha| < \beta < 2\alpha \right\}. \quad (1.9)$$

See Figure 2 in which S^o is shadowed, whose closure is S defined in (1.8).

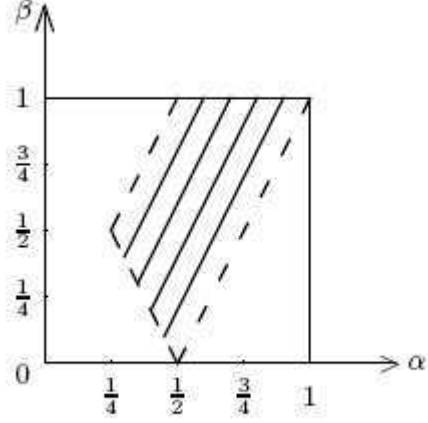


Figure 2: Region of C^∞ smoothness

Now, we divide the unit square $[0, 1] \times [0, 1]$ further as shown in Figure 3, where

$$\begin{cases} R_1 = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid \alpha \leq \beta \leq 2\alpha - \frac{1}{2} \right\}, \\ R_2 = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid \left(2\alpha - \frac{1}{2} \right) \vee \frac{1}{2} < \beta < 2\alpha \right\}, \\ R_3 = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid 0 \leq 1 - 2\alpha < \beta \leq \frac{1}{2}, (\alpha, \beta) \neq \left(\frac{1}{2}, \frac{1}{2} \right) \right\}, \\ R_4 = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid 0 < 2\alpha - 1 \leq \beta < \alpha \right\}, \\ R_5 = \left\{ (\alpha, \beta) \in [0, 1] \times [0, 1] \mid 0 < \beta < 2\alpha - 1 \right\}, \\ R_6 = ([0, 1] \times [0, 1]) \setminus (R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5) = S_1 \cup S_2 \cup S_I, \end{cases} \quad (1.10)$$

with $S_I = (\frac{1}{2}, 1] \times \{0\}$.

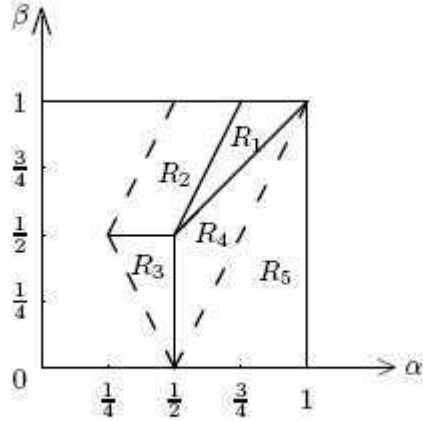


Figure 3: Region of regularity

We see that

$$S^o = R_1 \cup R_2 \cup R_3 \cup R_4, \quad S_3 = R_5 \cup S_I.$$

• In 1998, Liu and Yong obtained several regularity results for a general coupled system ([12]), which implies that the semigroup $e^{\mathcal{A}_{\alpha, \beta} t}$ is analytic in R_1 , and is of *Gevrey* class $\delta > \frac{1}{2(2\alpha - \beta)}$ in R_2 .

• In 2006, Denk and Racke showed that in region R_1 the semigroup remains analytic in Banach space $L^p(R^n)$, for all $1 < p < \infty$, with A being $-\Delta$ ([6]).

It is natural to ask what can we say about the regularity of the semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$ for all the values of $\alpha, \beta \in [0, 1]$, beyond just being analytic in R_1 and being C^∞ in S^o ? The main results of this paper can be stated as follows.

Theorem 1.2. *The semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$ has the following regularity properties:*

- (i) *It is analytic in R_1 ;*
- (ii) *It is of Gevrey class $\delta > \frac{1}{\mu(\alpha,\beta)}$ in $R_2 \cup R_3 \cup R_4 \cup R_5$ with*

$$\mu(\alpha, \beta) = \begin{cases} 2[(2\alpha - \beta) \wedge (2\alpha + \beta - 1)], & (\alpha, \beta) \in R_2 \cup R_3, \\ \frac{\beta}{\alpha}, & (\alpha, \beta) \in R_4 \cup R_5; \end{cases} \quad (1.11)$$

- (iii) *It is not differentiable in R_6 .*

Moreover, if A admits a sequence of eigenvalues $\mu_n \in \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} \mu_n = \infty, \quad (1.12)$$

then the Gevrey class orders in (ii) are sharp in the following sense: For any $\varepsilon > 0$, the semigroup is not Gevrey class of order $\frac{1}{\mu(\alpha,\beta)+\varepsilon}$.

The significance of the above result includes the following:

- In the region $R_2 \cup R_3 \cup R_4$, we establish that $e^{\mathcal{A}_{\alpha,\beta}t}$ is of proper order Gevrey classes, instead of just saying that it is C^∞ as in [15].
- The semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$ is also shown to be Gevrey class of a proper order in R_5 and not even differentiable in R_6 , where, to our best knowledge, there is no regularity result for the semigroup in the region $R_5 \cup R_6$ so far.
- The Gevrey class orders are sharp for the case that A is a positive definite self-adjoint operator having a sequence of (real) eigenvalues that goes to infinite. This is the case when A is a usual elliptic differential operator, say, $-\Delta$ in a bounded domain.

Note that

$$\frac{1}{\mu(\alpha, \beta)} = \begin{cases} \frac{1}{2(2\alpha - \beta)}, & (\alpha, \beta) \in R_2, \\ \frac{1}{2(2\alpha + \beta) - 2}, & (\alpha, \beta) \in R_3. \end{cases}$$

In a word, our results complete the regularity analysis on the semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$, in a certain sense. Combining our results with those found in the literature, we have the following summary:

Regions	Regularity	Stability
R_1	analytic	exponentially stable
R_2	Gevrey class $\delta > \frac{1}{2(2\alpha-\beta)}$	exponentially stable
R_3	Gevrey class $\delta > \frac{1}{2(2\alpha+\beta)-2}$	exponentially Stable
R_4	Gevrey class $\delta > \frac{\alpha}{\beta}$	exponentially stable
R_5	Gevrey class $\delta > \frac{\alpha}{\beta}$	not asymptotically stable
S_I	not differentiable	not asymptotically stable
$(S_1 \cup S_2) \cap S$	not differentiable	exponentially stable
S_1	not differentiable	polynomially stable of order $\frac{1}{2(\beta-2\alpha)}$
S_2	not differentiable	polynomially stable of order $\frac{1}{2-2(2\alpha+\beta)}$

The rest of the paper is organized as following. Sections 2 and 3 are devoted to the proof of (i)–(iii) of Theorem 1.2, showing that the semigroup $e^{\mathcal{A}_{\alpha,\beta}t}$ has proper regularity in different regions. Section 4 is for the asymptotic analysis on an eigenvalue sequence of $\mathcal{A}_{\alpha,\beta}$, assuming that A has a sequence of eigenvalues satisfying (1.12). Such an analysis will enable us to show that the orders of Gevrey class obtained in Sections 2 and 3 in different parts of the unit square are sharp.

2 Proof of the Main Result

For the simplicity of presentation, we will take $\gamma = k = 1$ throughout the rest of the paper.

In this section, we will present a proof for part (i)–(iii) of Theorem 1.2. To this end, let us first recall the following standard result which is stated in a comparable way (see [16, 13] for parts (i)–(ii), [18] for part (iii), [12] for (iv), and [2] for (v)).

Lemma 2.1. *Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ generate a C_0 -semigroup $e^{\mathcal{A}t}$ on \mathcal{H} such that*

$$\|e^{\mathcal{A}t}\| \leq M, \quad \forall t \geq 0, \quad (2.1)$$

for some $M \geq 1$ and

$$i\lambda \in \rho(\mathcal{A}), \quad \forall \lambda \in \mathbb{R}, \quad |\lambda| \text{ large enough.} \quad (2.2)$$

Then the following hold:

(i) Semigroup $e^{\mathcal{A}t}$ is analytic if and only if for some $a \in \mathbb{R}$ and $b, C > 0$ such that

$$\rho(\mathcal{A}) \supseteq \Sigma(a, b) \equiv \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > a - b|\operatorname{Im} \lambda| \right\}, \quad (2.3)$$

and

$$\|(i\lambda - \mathcal{A})^{-1}\| \leq \frac{C}{1 + |\lambda|}, \quad \lambda \in \Sigma(a, b). \quad (2.4)$$

This is the case if and only if

$$\overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} |\lambda| \|(i\lambda - \mathcal{A})^{-1}\| < \infty. \quad (2.5)$$

(ii) Semigroup $e^{\mathcal{A}t}$ is of Gevrey class $\delta > 1$ if and only if for any $b, \tau > 0$, there are constants $a \in \mathbb{R}$ and $C > 0$ depending on b, τ, δ such that

$$\rho(\mathcal{A}) \supseteq \Sigma_b(\delta) \equiv \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > a - b|\operatorname{Im} \lambda|^{\frac{1}{\delta}} \right\}, \quad (2.6)$$

and

$$\|(i\lambda - \mathcal{A})^{-1}\| \leq C \left(e^{-\tau \operatorname{Re} \lambda} + 1 \right), \quad \forall \lambda \in \Sigma_b(\delta). \quad (2.7)$$

This is the case, in particular, if for some $\mu \in (\delta^{-1}, 1)$,

$$\overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} |\lambda|^\mu \|(i\lambda - \mathcal{A})^{-1}\| < \infty. \quad (2.8)$$

(iii) Semigroup $e^{\mathcal{A}t}$ is differentiable if and only if for any $b > 0$, there are constants $a_b \in \mathbb{R}$ and $C_b > 0$ such that

$$\rho(\mathcal{A}) \supseteq \Sigma_b \equiv \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > a_b - b \log |\operatorname{Im} \lambda| \right\}, \quad (2.9)$$

and

$$\|(i\lambda - \mathcal{A})^{-1}\| \leq C_b |\operatorname{Im} \lambda|, \quad \forall \lambda \in \Sigma_b, \operatorname{Re} \lambda \leq 0. \quad (2.10)$$

This is the case, in particular, if

$$\overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \log |\lambda| \|(i\lambda - \mathcal{A})^{-1}\| = 0. \quad (2.11)$$

(iv) **(Gearhart–Pruss)** Semigroup $e^{\mathcal{A}t}$ is exponentially stable if and only if

$$i\lambda \in \rho(\mathcal{A}), \quad \forall \lambda \in \mathbb{R}, \quad (2.12)$$

and

$$\overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \|(i\lambda - \mathcal{A})^{-1}\| < \infty. \quad (2.13)$$

(v) **(Borichev–Tomilov)** Semigroup $e^{\mathcal{A}t}$ is polynomially stable of order $j > 0$ if and only if (2.5) holds and

$$\overline{\lim}_{|\lambda| \rightarrow \infty} |\lambda|^{-\frac{1}{j}} \|(i\lambda - \mathcal{A})^{-1}\| < \infty. \quad (2.14)$$

For notational simplicity, hereafter, we write $i\lambda - \mathcal{A}$ instead of $i\lambda I - \mathcal{A}$, omitting I . In the above result, the regularity and stability properties of the semigroup $e^{\mathcal{A}t}$ are deliberately related to the spectral/resolvent of the generator \mathcal{A} . Practically, we will use the limit relations (2.5), (2.8) and (2.11) to establish the regularity property of the semigroup, and use the spectrum relations (2.3), (2.6) and (2.9) to show that the relevant indices are sharp. The following corollary will be useful below.

Corollary 2.2. (i) Suppose $\sigma(\mathcal{A})$ contains a sequence λ_n such that

$$\lim_{n \rightarrow \infty} \operatorname{Re} \lambda_n = a, \quad \lim_{n \rightarrow \infty} |\lambda_n| = \infty, \quad (2.15)$$

for some $a \in \mathbb{R}$, then the semigroup $e^{\mathcal{A}t}$ is not differentiable.

(ii) Suppose there exists a sequence $\lambda_n \in \sigma(\mathcal{A})$ such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{\operatorname{Re} \lambda_n}{|\operatorname{Im} \lambda_n|^{\frac{1}{\delta}}} = 0. \quad (2.16)$$

Then $e^{\mathcal{A}t}$ is not of Gevrey class δ .

Proof. (i) Suppose $e^{\mathcal{A}t}$ is differentiable. Then for any $b > 0$, there exists an $a_b \in \mathbb{R}$ such that

$$\operatorname{Re} \lambda_n \leq a_b - b \log |\operatorname{Im} \lambda_n|, \quad n \geq 1,$$

since $\lambda_n \in \sigma(\mathcal{A})$. Letting $n \rightarrow \infty$ will lead to a contradiction. Hence the semigroup $e^{\mathcal{A}t}$ is not differentiable.

(ii) We use part (ii) of Lemma 2.1. Suppose $e^{\mathcal{A}t}$ is of Gevrey class $\delta > 0$, then from (2.6), for any $b > 0$, there exists an $a \in \mathbb{R}$ such that

$$\operatorname{Re} \lambda_n \leq a - b |\operatorname{Im} \lambda_n|^{\frac{1}{\delta}}, \quad \forall n \geq 1,$$

since $\lambda_n \in \sigma(\mathcal{A})$. Thus,

$$0 = \overline{\lim}_{n \rightarrow \infty} \frac{\operatorname{Re} \lambda_n}{|\operatorname{Im} \lambda_n|^{\frac{1}{\delta}}} \leq -b,$$

a contradiction. □

We now state two results whose proof will be carried out in the following section.

Theorem 2.3. *Let*

$$\mu(\alpha, \beta) = \begin{cases} 1, & (\alpha, \beta) \in R_1, \\ 2[(2\alpha - \beta) \wedge (2\alpha + \beta - 1)], & (\alpha, \beta) \in R_2 \cup R_3 \cup S_1 \cup S_2, \\ \frac{\beta}{\alpha}, & (\alpha, \beta) \in R_4 \cup R_5 \cup S_I. \end{cases} \quad (2.17)$$

Then

$$\overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} |\lambda|^{\mu(\alpha, \beta)} \|(i\lambda - \mathcal{A}_{\alpha, \beta})^{-1}\| < \infty. \quad (2.18)$$

Theorem 2.4. *Let A admit a sequence of eigenvalues $\mu_n \in \mathbb{R}$ such that*

$$\lim_{n \rightarrow \infty} \mu_n = \infty.$$

Then there exists a sequence $\lambda_n \in \sigma(\mathcal{A}_{\alpha, \beta})$ of eigenvalues of $\mathcal{A}_{\alpha, \beta}$ such that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{\operatorname{Re} \lambda_n}{|\operatorname{Im} \lambda_n|^{\mu(\alpha, \beta) + \varepsilon}} = 0, \quad \forall (\alpha, \beta) \in R_2 \cup R_3 \cup R_4 \cup R_5, \quad (2.19)$$

and

$$\lim_{n \rightarrow \infty} \operatorname{Re} \lambda_n = a, \quad \lim_{n \rightarrow \infty} |\lambda_n| = \infty, \quad \forall (\alpha, \beta) \in R_6. \quad (2.20)$$

To close this section we present a proof of Theorem 1.2.

Proof of Theorem 1.2. Combining Theorem 2.3 and Lemma 2.1, we obtain that the semigroup $e^{\mathcal{A}_{\alpha, \beta}t}$ is analytic in R_1 , is Gevrey class of order $\delta > \frac{1}{\mu(\alpha, \beta)}$ in $R_2 \cup R_3 \cup R_4 \cup R_5$. Also, in $R_6 \equiv S_1 \cup S_2 \cup S_I$, (2.20) holds. Hence, by Corollary 2.2, the semigroup $e^{\mathcal{A}_{\alpha, \beta}t}$ is not differentiable there.

Next, by (2.19) and Corollary 2.2, we see that the Gevrey class order $\delta > \frac{1}{\mu(\alpha, \beta)}$ of the semigroup for $(\alpha, \beta) \in R_2 \cup R_3 \cup R_4 \cup R_5$ is sharp. □

We note that

$$\mu(\alpha, \beta) = 2(2\alpha - \beta) < 0, \quad (\alpha, \beta) \in S_1,$$

and

$$\mu(\alpha, \beta) = 2(\beta + 2\alpha) - 2 < 0, \quad (\alpha, \beta) \in S_2.$$

Thus, the corresponding (2.18) implies that the semigroup $e^{\mathcal{A}_{\alpha, \beta}t}$ is polynomially stable with order $\frac{1}{2(\beta - 2\alpha)}$ and $\frac{1}{2 - 2(\beta + 2\alpha)}$, respectively. The above two cases are exactly those found in [11].

3 Analysis on the Resolvent

In this section, we will prove Theorem 2.3. It is technical and lengthy. Let us now make some preparations. First of all, in our proof, the following interpolation theorem will play a crucial role.

Lemma 3.1. *Let $A : \mathcal{D}(A) \subseteq H$ be self-adjoint and positive definite. Then*

$$\|A^p x\| \leq \|A^q x\|^{\frac{p-r}{q-r}} \|A^r x\|^{\frac{q-p}{q-r}}, \quad \forall 0 \leq r \leq p \leq q, \quad x \in \mathcal{D}(A^q). \quad (3.1)$$

In particular, for any $\theta \in [0, \frac{1}{2}]$, one has (with $r = 0$, $p = \theta$, and $q = \frac{1}{2}$)

$$\|A^\theta x\| \leq \|A^{\frac{1}{2}} x\|^{2\theta} \|x\|^{1-2\theta}, \quad \forall x \in \mathcal{D}(A^{\frac{1}{2}}), \quad (3.2)$$

and for any $\theta \in [\frac{1}{2}, 1]$ (with $r = \frac{1}{2}$, $p = \theta$, and $q = 1$)

$$\|A^\theta x\| \leq \|Ax\|^{2\theta-1} \|A^{\frac{1}{2}} x\|^{2(1-\theta)}, \quad \forall x \in \mathcal{D}(A). \quad (3.3)$$

The above result is standard. For reader's convenience, we give a proof here which is very simple and it just costs us a few lines.

Proof. Since A is self-adjoint and positive definite, it admits a spectrum decomposition. More precisely, there exists a family of orthogonal projection operators $\{\mathbb{E}_\lambda, \lambda \in \sigma(A)\}$, with $\lambda \mapsto \mathbb{E}_\lambda$ being nondecreasing such that

$$Ax = \int_{\sigma(A)} \lambda d\mathbb{E}_\lambda x, \quad \forall x \in \mathcal{D}(A), \quad (3.4)$$

where $\sigma(A) \subseteq (0, \infty)$ is the spectrum of A . Clearly, for any $\theta \in \mathbb{R}$,

$$A^\theta x = \int_{\sigma(A)} \lambda^\theta d\mathbb{E}_\lambda x, \quad x \in \mathcal{D}(A^\theta). \quad (3.5)$$

Now, for any $0 \leq r \leq p \leq q$, $x \in \mathcal{D}(A^q)$, by Hölder's inequality, one has

$$\begin{aligned} \|A^p x\|^2 &= \int_{\sigma(A)} \lambda^{2p} d\|\mathbb{E}_\lambda x\|^2 \leq \left(\int_{\sigma(A)} \lambda^{2q} d\|\mathbb{E}_\lambda x\|^2 \right)^{\frac{p-r}{q-r}} \left(\int_{\sigma(A)} \lambda^{2r} d\|\mathbb{E}_\lambda x\|^2 \right)^{\frac{q-p}{q-r}} \\ &= \|A^q x\|^{\frac{2(p-r)}{q-r}} \|A^r x\|^{\frac{2(q-p)}{q-r}}. \end{aligned}$$

This proves (3.1). The two special cases (3.2) and (3.3) are clear from (3.1). □

Next, for any $\lambda \in \mathbb{R}$, and any $U \equiv (u, v, w)^T \in \mathcal{D}(\mathcal{A}_{\alpha, \beta})$,

$$(i\lambda - \mathcal{A}_{\alpha, \beta})U = \begin{pmatrix} i\lambda & -I & 0 \\ A & i\lambda & -A^\alpha \\ 0 & A^\alpha & i\lambda + A^\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} i\lambda u - v \\ Au + i\lambda v - A^\alpha w \\ A^\alpha v + (i\lambda + A^\beta)w \end{pmatrix}. \quad (3.6)$$

Our proof for Theorem 2.3 will be based on a contradiction argument. Suppose for some given $(\alpha, \beta, \mu) \in [0, 1] \times [0, 1] \times [0, 1]$, without having any specific relations among them, the following is not true:

$$\overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} |\lambda|^\mu \|(i\lambda - \mathcal{A}_{\alpha, \beta})^{-1}\| < \infty.$$

Then there exists a sequence $\{(\lambda_n, U_n) \mid n \geq 1\} \subseteq \mathbb{R} \times \mathcal{D}(\mathcal{A}_{\alpha, \beta})$ with $U_n \equiv (u_n, v_n, w_n)^T$, and

$$\begin{cases} \lim_{n \rightarrow \infty} |\lambda_n| = \infty, \\ \|U_n\|_{\mathcal{H}}^2 = \|A^{\frac{1}{2}}u_n\|^2 + \|v_n\|^2 + \|w_n\|^2 = 1, \quad n \geq 1, \end{cases} \quad (3.7)$$

such that

$$\lim_{n \rightarrow \infty} |\lambda_n|^{-\mu} \|(i\lambda_n - \mathcal{A}_{\alpha, \beta})U_n\|_{\mathcal{H}} = 0, \quad (3.8)$$

i.e. (note (3.6))

$$i\lambda_n |\lambda_n|^{-\mu} A^{\frac{1}{2}}u_n - |\lambda_n|^{-\mu} A^{\frac{1}{2}}v_n = o(1), \quad (3.9a)$$

$$i\lambda_n |\lambda_n|^{-\mu} v_n + |\lambda_n|^{-\mu} Au_n - |\lambda_n|^{-\mu} A^\alpha w_n = o(1), \quad (3.9b)$$

$$i\lambda_n |\lambda_n|^{-\mu} w_n + |\lambda_n|^{-\mu} A^\alpha v_n + |\lambda_n|^{-\mu} A^\beta w_n = o(1). \quad (3.9c)$$

Hereafter $o(1)$ stands for a vector in H (or a quantity in \mathbb{R}) which goes to zero as $n \rightarrow \infty$. The advantage of using such a notation is that (3.9a)–(3.9c) can be regarded as a system of equations, which will be convenient below. For the sequence $\{(\lambda_n, u_n, v_n, w_n)\}$ satisfying (3.9a)–(3.9c), we have the following result.

Lemma 3.2. *The following is true:*

$$i\lambda_n |\lambda_n|^{-\mu} \|A^{\frac{1}{2}}u_n\|^2 - |\lambda_n|^{-\mu} \langle v_n, Au_n \rangle = o(1), \quad (3.10a)$$

$$i\lambda_n |\lambda_n|^{-\mu} \|v_n\|^2 + |\lambda_n|^{-\mu} \langle Au_n, v_n \rangle - |\lambda_n|^{-\mu} \langle A^\alpha w_n, v_n \rangle = o(1), \quad (3.10b)$$

$$i\lambda_n |\lambda_n|^{-\mu} \|w_n\|^2 + |\lambda_n|^{-\mu} \langle A^\alpha v_n, w_n \rangle = o(1), \quad (3.10c)$$

$$|\lambda_n|^{-\mu} \|A^{\frac{\beta}{2}}w_n\|^2 = o(1), \quad (3.10d)$$

$$\|A^{\frac{1}{2}}u_n\|^2 + \|w_n\|^2 = \frac{1}{2} + o(1), \quad (3.10e)$$

$$\|v_n\|^2 = \frac{1}{2} + o(1), \quad (3.10f)$$

$$|\lambda_n|^{-1} \|A^{\frac{1}{2}}v_n\| + |\lambda_n|^{-1} \|Au_n - A^\alpha w_n\| + |\lambda_n|^{-1} \|A^\alpha v_n + A^\beta w_n\| = O(1). \quad (3.10g)$$

Hereafter, $O(1)$ stands for a bounded quantity (uniformly in $n \geq 1$) in \mathbb{R} .

Proof. By taking inner products of (3.9a) with $A^{\frac{1}{2}}u_n$ and (3.9b) with v_n , respectively, we obtain (3.10a) and (3.10b). Next, by taking inner product of (3.9c) with w_n , we have

$$i\lambda_n |\lambda_n|^{-\mu} \|w_n\|^2 + |\lambda_n|^{-\mu} \|A^{\frac{\beta}{2}}w_n\|^2 + |\lambda_n|^{-\mu} \langle A^\alpha v_n, w_n \rangle = o(1). \quad (3.11)$$

Adding the obtained (3.10a) and (3.10b) to (3.11), one has

$$\begin{aligned} & |\lambda_n|^{-\mu} \|A^{\frac{\beta}{2}}w_n\|^2 + i \left[|\lambda_n| |\lambda_n|^{-\mu} \left(\|A^{\frac{1}{2}}u_n\|^2 + \|v_n\|^2 + \|w_n\|^2 \right) \right. \\ & \quad \left. + 2|\lambda_n|^{-\mu} \left(\operatorname{Im} \langle Au_n, v_n \rangle + \operatorname{Im} \langle A^\alpha v_n, w_n \rangle \right) \right] \\ & = |\lambda_n|^{-\mu} \|A^{\frac{\beta}{2}}w_n\|^2 + i \left[|\lambda_n| |\lambda_n|^{-\mu} + 2|\lambda_n|^{-\mu} \left(\operatorname{Im} \langle Au_n, v_n \rangle + \operatorname{Im} \langle A^\alpha v_n, w_n \rangle \right) \right] = o(1). \end{aligned}$$

Thus, (3.10d) follows. Thanks to this equation, (3.10c) follows from (3.11).

On the other hand, by taking conjugate of (3.10a) and then multiplying it by (-1) , we have

$$i\lambda_n|\lambda_n|^{-\mu}\|A^{\frac{1}{2}}u_n\|^2 + |\lambda_n|^{-\mu}\langle Au_n, v_n \rangle = o(1). \quad (3.12)$$

By taking conjugate of (3.10c) and then multiplying it by (-1) , we have

$$i\lambda_n|\lambda_n|^{-\mu}\|w_n\|^2 - |\lambda_n|^{-\mu}\langle A^\alpha w_n, v_n \rangle = o(1). \quad (3.13)$$

Combining (3.10b) with (3.12)–(3.13), one obtains

$$i\lambda_n|\lambda_n|^{-\mu}\left(\|A^{\frac{1}{2}}u_n\|^2 - \|v_n\|^2 + \|w_n\|^2\right) = o(1), \quad (3.14)$$

leading to

$$\|A^{\frac{1}{2}}u_n\|^2 - \|v_n\|^2 + \|w_n\|^2 = o(1). \quad (3.15)$$

Taking into account $\|U_n\|_{\mathcal{H}}^2 = 1$, we obtain (3.10e)–(3.10f). Finally, by dividing (3.9a)–(3.9c) by $\lambda_n|\lambda_n|^{-\mu}$ (note $\mu \leq 1$), one has

$$iA^{\frac{1}{2}}u_n - \lambda_n^{-1}A^{\frac{1}{2}}v_n = o(1), \quad (3.16a)$$

$$iv_n + \lambda_n^{-1}Au_n - \lambda_n^{-1}A^\alpha w_n = o(1), \quad (3.16b)$$

$$iw_n + \lambda_n^{-1}A^\alpha v_n + \lambda_n^{-1}A^\beta w_n = o(1), \quad (3.16c)$$

which implies (3.10g). \square

In what follows, for specific situations, we will end up with

$$\text{either } \|A^{\frac{1}{2}}u_n\|^2 + \|w_n\|^2 = o(1), \quad \text{or } \|v_n\|^2 = o(1),$$

to lead to a contradiction. Now, we present a detailed proof for Theorem 2.3.

Proof of Theorem 2.3. The proof for $(\alpha, \beta) \in S_1 \cup S_2$ can be found in [11]. We carry out the proof for the rest parts of the regions in $[0, 1] \times [0, 1]$. We divide the proof into several cases.

Case 1. Let $(\alpha, \beta) \in R_1$, i.e.,

$$\alpha \leq \beta \leq 2\alpha - \frac{1}{2}, \quad \mu = 1. \quad (3.17)$$

In this case, (3.9a)–(3.9c) are equivalent to (3.16a)–(3.16c). Since $\alpha \leq \beta$, $A^{\alpha-\beta}$ is bounded. Applying this bounded operator to (3.16c), we get

$$iA^{\alpha-\beta}w_n + \lambda_n^{-1}A^{2\alpha-\beta}v_n + \lambda_n^{-1}A^\alpha w_n = o(1). \quad (3.18)$$

Adding the above to (3.16b), we obtain

$$iv_n + \lambda_n^{-1}Au_n + iA^{\alpha-\beta}w_n + \lambda_n^{-1}A^{2\alpha-\beta}v_n = o(1). \quad (3.19)$$

Furthermore, taking inner product of the above with v_n yields,

$$i\|v_n\|^2 + \lambda_n^{-1}\langle A^{\frac{1}{2}}u_n, A^{\frac{1}{2}}v_n \rangle + i\langle A^{\alpha-\beta}w_n, v_n \rangle + \lambda_n^{-1}\|A^{\alpha-\frac{\beta}{2}}v_n\|^2 = o(1). \quad (3.20)$$

The first and the third terms in the above are clearly bounded. Making use of (3.10g), we see that the second term in the above is also bounded. So is the fourth term:

$$|\lambda_n|^{-1}\|A^{\alpha-\frac{\beta}{2}}v_n\|^2 = O(1). \quad (3.21)$$

Thus, noting $\mu = 1$, and using (3.10d), we have

$$\begin{aligned} |\lambda_n|^{-1} |\langle A^\alpha v_n, w_n \rangle| &= |\lambda_n|^{-1} |\langle A^{\alpha-\frac{\beta}{2}} v_n, A^{\frac{\beta}{2}} w_n \rangle| \\ &\leq (|\lambda_n|^{-\frac{1}{2}} \|A^{\alpha-\frac{\beta}{2}} v_n\|) (|\lambda_n|^{-\frac{\mu}{2}} \|A^{\frac{\beta}{2}} w_n\|) = o(1). \end{aligned}$$

Then (3.10c) implies

$$\|w_n\| = o(1), \quad (3.22)$$

and (3.10b) becomes

$$i\|v_n\|^2 + \lambda_n^{-1} \langle Au_n, v_n \rangle = o(1). \quad (3.23)$$

Also, since $\alpha \leq \beta$, (3.20) implies

$$i\|v_n\|^2 + \lambda_n^{-1} \langle Au_n, v_n \rangle + \lambda_n^{-1} \|A^{\alpha-\frac{\beta}{2}} v_n\|^2 = o(1). \quad (3.24)$$

Combining (3.23)–(3.24), one gets

$$\lambda_n^{-1} \|A^{\alpha-\frac{\beta}{2}} v_n\|^2 = o(1), \quad (3.25)$$

which improves (3.21). Moreover, since $\alpha \leq \beta$, by (3.22), we may write (3.19) as

$$iv_n + \lambda_n^{-1} Au_n + \lambda_n^{-1} A^{2\alpha-\beta} v_n = o(1). \quad (3.26)$$

Further, since $\frac{1}{2} \leq 2\alpha - \beta \leq 1$, $\|A^{1-(2\alpha-\beta)} u_n\|$ is bounded. Taking inner product (3.26) with $A^{1-(2\alpha-\beta)} u_n$ in H , we obtain

$$i \langle v_n, A^{1-(2\alpha-\beta)} u_n \rangle + \langle \lambda_n^{-1} A^{\frac{1}{2}} v_n, A^{\frac{1}{2}} u_n \rangle + |\lambda_n|^{-1} \|A^{1-(\alpha-\frac{\beta}{2})} u_n\|^2 = o(1).$$

The first two terms in the above are bounded. So is the third term. Therefore, making use of (3.25), we finally obtain

$$|\lambda_n|^{-1} |\langle Au_n, v_n \rangle| = (|\lambda_n|^{-\frac{1}{2}} \|A^{1-(\alpha-\frac{\beta}{2})} u_n\|) (|\lambda_n|^{-\frac{1}{2}} \|A^{\alpha-\frac{\beta}{2}} v_n\|) = o(1).$$

Then (3.23) implies

$$\|v_n\|^2 = o(1), \quad (3.27)$$

which is a contradiction to (3.10f).

Case 2. Let $(\alpha, \beta) \in R_2$, i.e.,

$$\left(2\alpha - \frac{1}{2}\right) \vee \frac{1}{2} < \beta < 2\alpha, \quad \mu \equiv \mu(\alpha, \beta) = 2(2\alpha - \beta). \quad (3.28)$$

Note that (see Fig.2) in the current case,

$$\alpha < \beta. \quad (3.29)$$

From (3.28), one has

$$0 < \frac{\mu}{4} = \alpha - \frac{\beta}{2} = \frac{1}{2} - \frac{1}{2}(\beta - 2\alpha + 1) < \frac{1}{2}.$$

Thus, by interpolation, using (3.10d) and (3.10g), we have

$$\begin{aligned} |\lambda_n|^{-\mu} |\langle A^\alpha v_n, w_n \rangle| &\leq |\lambda_n|^{-\frac{\mu}{2}} \|A^{\alpha-\frac{\beta}{2}} v_n\| (|\lambda_n|^{-\frac{\mu}{2}} \|A^{\frac{\beta}{2}} w_n\|) \\ &\leq |\lambda_n|^{-(2\alpha-\beta)} \|A^{\frac{1}{2}} v_n\|^{2\alpha-\beta} \|v_n\|^{1-2\alpha+\beta} o(1) \leq (|\lambda_n|^{-1} \|A^{\frac{1}{2}} v_n\|)^{2\alpha-\beta} o(1) = o(1). \end{aligned}$$

Consequently, (3.10c) can be written as

$$\lambda_n |\lambda_n|^{-\mu} \|w_n\|^2 = o(1), \quad (3.30)$$

and (3.10b) can be written as

$$i\lambda_n |\lambda_n|^{-\mu} \|v_n\|^2 + |\lambda_n|^{-\mu} \langle Au_n, v_n \rangle = o(1), \quad (3.31)$$

which implies

$$i\|v_n\|^2 + \lambda_n^{-1} \langle Au_n, v_n \rangle = o(1). \quad (3.32)$$

We now show that

$$\lambda_n^{-1} \langle Au_n, v_n \rangle = o(1). \quad (3.33)$$

Since $\alpha < \beta$, $A^{\alpha-\beta}$ is bounded. Applying $A^{\alpha-\beta}$ to (3.9c), we have

$$i\lambda_n |\lambda_n|^{-\mu} A^{\alpha-\beta} w_n + |\lambda_n|^{-\mu} A^{2\alpha-\beta} v_n + |\lambda_n|^{-\mu} A^\alpha w_n = o(1).$$

Adding the above to (3.9b), one has

$$i\lambda_n |\lambda_n|^{-\mu} v_n + |\lambda_n|^{-\mu} Au_n + |\lambda_n|^{-\mu} A^{2\alpha-\beta} v_n + i\lambda_n |\lambda_n|^{-\mu} A^{\alpha-\beta} w_n = o(1), \quad (3.34)$$

which implies

$$iv_n + \lambda_n^{-1} Au_n + \lambda_n^{-1} A^{2\alpha-\beta} v_n + iA^{\alpha-\beta} w_n = o(1). \quad (3.35)$$

Further, by (3.30), $\|w_n\| = o(1)$. Thus, the above becomes

$$iv_n + \lambda_n^{-1} Au_n + \lambda_n^{-1} A^{2\alpha-\beta} v_n = o(1). \quad (3.36)$$

By (3.10g), for any $0 \leq \nu \leq \frac{1}{2}$,

$$|\lambda_n|^{-2\nu} \|A^\nu v_n\| \leq |\lambda_n|^{-2\nu} \|A^{\frac{1}{2}} v_n\|^{2\nu} \|v_n\|^{1-2\nu} \leq (|\lambda_n|^{-1} \|A^{\frac{1}{2}} v_n\|)^{2\nu} = O(1). \quad (3.37)$$

Since $2\alpha - \beta = \frac{\mu}{2} < \frac{1}{2}$, one has

$$|\lambda_n|^{-1} \|A^{2\alpha-\beta} v_n\| = |\lambda_n|^{\mu-1} (|\lambda_n|^{-\mu} \|A^{\frac{\mu}{2}} v_n\|) = |\lambda_n|^{-(1-\mu)} O(1) = o(1).$$

Thus, (3.36) becomes

$$iv_n + \lambda_n^{-1} Au_n = o(1). \quad (3.38)$$

Consequently, we obtain

$$|\lambda_n|^{-1} \|Au_n\| = O(1). \quad (3.39)$$

By interpolation, together with (3.39)

$$|\lambda_n|^{-\frac{1+\mu}{2}} \|A^{\frac{3+\mu}{4}} u_n\| = |\lambda_n|^{-\frac{1+\mu}{2}} \|A^{\frac{1+\mu}{4}} (A^{\frac{1}{2}} u_n)\| \leq \|\lambda_n^{-1} Au_n\|^{\frac{1+\mu}{2}} \|A^{\frac{1}{2}} u_n\|^{\frac{1-\mu}{2}} = O(1).$$

Now, taking inner product of (3.34) with $|\lambda_n|^{-\frac{1+\mu}{2}} A^{\frac{1+\mu}{4}} v_n$ in H leads to

$$\begin{aligned} & i\lambda_n |\lambda_n|^{-\frac{1+3\mu}{2}} \|A^{\frac{1+\mu}{8}} v_n\|^2 + |\lambda_n|^{-\frac{1+3\mu}{2}} \langle Au_n, A^{\frac{1+\mu}{4}} v_n \rangle \\ & + |\lambda_n|^{-\frac{1+3\mu}{2}} \|A^{\frac{1+3\mu}{8}} v_n\|^2 + i\lambda_n |\lambda_n|^{-\frac{1+3\mu}{2}} \langle A^{\alpha-\beta} w_n, A^{\frac{1+\mu}{4}} v_n \rangle = o(1). \end{aligned} \quad (3.40)$$

Recall that $\beta \in (\frac{1}{2}, 1)$ and $\mu = 2(2\alpha - \beta)$. Thus,

$$\alpha - \beta + \frac{1+\mu}{4} = \frac{\mu}{4} - \frac{\beta}{2} + \frac{1+\mu}{4} = \frac{2\mu+1-2\beta}{4} < \frac{\mu}{2} < \frac{1}{2},$$

and

$$\frac{2-\mu-2\beta}{2} = \frac{1-\mu}{2} + \frac{1-2\beta}{2} < \frac{1-\mu}{2}.$$

Hence, it follow from (3.30) that

$$\begin{aligned} |\lambda_n|^{\frac{1-3\mu}{2}} |\langle A^{\alpha-\beta} w_n, A^{\frac{1+\mu}{4}} v_n \rangle| &= |\lambda_n|^{\frac{2-\mu-2\beta}{2}} \|w_n\| (|\lambda_n|^{-\frac{2\mu+1-2\beta}{2}} \|A^{\frac{2\mu+1-2\beta}{4}} v_n\|) \\ &\leq |\lambda_n|^{\frac{1-\mu}{2}} \|w_n\| O(1) = o(1). \end{aligned}$$

Then (3.40) becomes

$$i\lambda_n |\lambda_n|^{-\frac{1+3\mu}{2}} \|A^{\frac{1+\mu}{8}} v_n\|^2 + |\lambda_n|^{-\frac{1+3\mu}{2}} \langle Au_n, A^{\frac{1+\mu}{4}} v_n \rangle + |\lambda_n|^{-\frac{1+3\mu}{2}} \|A^{\frac{1+3\mu}{8}} v_n\|^2 = o(1). \quad (3.41)$$

Now, taking inner product of (3.9a) with $|\lambda_n|^{-\frac{1+\mu}{2}} A^{\frac{3+\mu}{4}} u_n$, we have

$$\begin{aligned} o(1) &= i\lambda_n |\lambda_n|^{-\frac{1+3\mu}{2}} \langle A^{\frac{1}{2}} u_n, A^{\frac{3+\mu}{4}} u_n \rangle - |\lambda_n|^{-\frac{1+3\mu}{2}} \langle A^{\frac{1}{2}} v_n, A^{\frac{3+\mu}{4}} u_n \rangle \\ &= i\lambda_n |\lambda_n|^{-\frac{1+3\mu}{2}} \|A^{\frac{5+\mu}{8}} u_n\|^2 - |\lambda_n|^{\frac{1+3\mu}{2}} \langle A^{\frac{1+\mu}{4}} v_n, Au_n \rangle. \end{aligned} \quad (3.42)$$

Adding (3.41) to (3.42) and taking its real part, we get

$$|\lambda_n|^{-\frac{1+3\mu}{2}} \|A^{\frac{1+3\mu}{8}} v_n\|^2 = o(1). \quad (3.43)$$

Consequently,

$$\begin{aligned} |\lambda_n|^{-1} |\langle Au_n, v_n \rangle| &\leq (|\lambda_n|^{-\frac{3(1-\mu)}{4}} \|A^{\frac{7-3\mu}{8}} u_n\|) (|\lambda_n|^{-\frac{1+3\mu}{4}} \|A^{\frac{1+3\mu}{8}} v_n\|) \\ &= |\lambda_n|^{-\frac{3(1-\mu)}{4}} \|A^{\frac{3(1-\mu)}{8}} (A^{\frac{1}{2}} u_n)\| o(1) \leq (|\lambda_n|^{-1} \|Au_n\|)^{\frac{3(1-\mu)}{4}} \|A^{\frac{1}{2}} u_n\|^{1-\frac{3(1-\mu)}{4}} o(1) = o(1). \end{aligned}$$

Thus, by (3.32), one obtains

$$\|v_n\| = o(1),$$

a contradiction to (3.10f) again.

Case 3. Let $(\alpha, \beta) \in R_3$, i.e.,

$$0 \leq 1 - 2\alpha < \beta \leq \frac{1}{2}, \quad \alpha \leq \frac{1}{2}, \quad \mu \equiv \mu(\alpha, \beta) = 2(2\alpha + \beta) - 2. \quad (3.44)$$

Multiplying the (3.10c) by $\lambda_n^{-1} |\lambda_n|^{2\beta}$, we get

$$i|\lambda_n|^{-\mu+2\beta} \|w_n\|^2 + \lambda_n^{-1} |\lambda_n|^{-\mu+2\beta} \langle A^\alpha v_n, w_n \rangle = o(1). \quad (3.45)$$

By (3.37) and (3.10d),

$$\begin{aligned} |\lambda_n|^{-1-\mu+2\beta} |\langle A^\alpha v_n, w_n \rangle| &\leq |\lambda_n|^{\frac{\mu}{2}+1-4\alpha} \|A^{\alpha-\frac{\beta}{2}} v_n\| (|\lambda_n|^{-\frac{\mu}{2}} \|A^{\frac{\beta}{2}} w_n\|) \\ &\leq (|\lambda_n|^{-(2\alpha-\beta)} \|A^{\alpha-\frac{\beta}{2}} v_n\|) (|\lambda_n|^{-\frac{\mu}{2}} \|A^{\frac{\beta}{2}} w_n\|) = o(1). \end{aligned} \quad (3.46)$$

Then we obtain from (3.45) that

$$|\lambda_n|^{2-4\alpha}\|w_n\|^2 = o(1), \quad (3.47)$$

which implies

$$\|w_n\| = o(1). \quad (3.48)$$

Next, applying bounded operator $A^{\alpha-\frac{1}{2}}$ to the first equation in (3.9a), we have

$$i\lambda_n|\lambda_n|^{-\mu}A^\alpha u_n - |\lambda_n|^{-\mu}A^\alpha v_n = o(1). \quad (3.49)$$

This allows us to rewrite (3.9c) as

$$i\lambda_n|\lambda_n|^{-\mu}w_n + i\lambda_n|\lambda_n|^{-\mu}A^\alpha u_n + |\lambda_n|^{-\mu}A^\beta w_n = o(1). \quad (3.50)$$

Note that for any $\nu \in [0, 1]$, $\frac{\nu+1}{2} \in [\frac{1}{2}, 1]$. Hence, by interpolation, we have

$$|\lambda_n|^{-\nu}\|A^{\frac{\nu+1}{2}}(u_n - A^{\alpha-1}w_n)\| \leq |\lambda_n|^{-\nu}\|A(u_n - A^{\alpha-1}w_n)\|^\nu\|A^{\frac{1}{2}}(u_n - A^{\alpha-1}w_n)\|^{1-\nu} = O(1) \quad (3.51)$$

due to (3.10g) and $\alpha \leq \frac{1}{2}$. By taking $\nu = 1 - 2\alpha \in [0, \frac{1}{2}]$, we obtain

$$\begin{aligned} |\lambda_n|^{2\alpha-1}\|A^{1-\alpha}u_n - w_n\| &= |\lambda_n|^{-\nu}\|A^{\frac{\nu+1}{2}}(u_n - A^{\alpha-1}w_n)\| \\ &\leq \|\lambda_n^{-1}(Au_n - A^\alpha w_n)\|^{1-2\alpha}\|A^{\frac{1}{2}}u_n - A^{\alpha-\frac{1}{2}}w_n\|^{2\alpha} = O(1). \end{aligned} \quad (3.52)$$

Since $\alpha + \beta < 1$ in R_3 which leads to $\mu = 2(\beta + 2\alpha) - 2 < 2\alpha$. Hence, $\mu - 1 < 2\alpha - 1$. We now take the inner product of (3.50) with $\lambda_n^{-1}|\lambda_n|^\mu(A^{1-\alpha}u_n - w_n)$ in H ,

$$i\langle w_n, A^{1-\alpha}u_n - w_n \rangle + i\|A^{\frac{1}{2}}u_n\|^2 - i\langle A^\alpha u_n, w_n \rangle + \lambda_n^{-1}\langle A^\beta w_n, A^{1-\alpha}u_n - w_n \rangle = o(1). \quad (3.53)$$

Observe that

$$|\langle w_n, A^{1-\alpha}u_n - w_n \rangle| = (|\lambda_n|^{1-2\alpha}\|w_n\|)(|\lambda_n|^{2\alpha-1}\|A^{1-\alpha}u_n - w_n\|) = o(1),$$

due to (3.47) and (3.52). It is obvious that the third term in (3.53) is an $o(1)$ because of $\alpha \leq \frac{1}{2}$. Furthermore, since $1 - 2\alpha + \beta \in (0, 1)$ in R_3 , we take $\nu = 1 - 2\alpha + \beta$ in (3.51) to obtain

$$|\lambda_n|^{-1+2\alpha-\beta}\|A^{1-\alpha+\frac{\beta}{2}}(u_n - A^{\alpha-1}w_n)\| = O(1).$$

Combining this estimate with (3.10d) and the fact $2\beta \leq 1$, we get

$$\begin{aligned} &|\lambda_n|^{-1}|\langle A^\beta w_n, A^{1-\alpha}u_n - w_n \rangle| \\ &\leq |\lambda_n|^{-1+2\beta}(|\lambda_n|^{-\frac{\mu}{2}}\|A^{\frac{\beta}{2}}w_n\|)(|\lambda_n|^{-1+2\alpha-\beta}\|A^{1-2\alpha+\frac{\beta}{2}}(u_n - A^{\alpha-1}w_n)\|) = o(1), \end{aligned}$$

i.e., the fourth term in (3.53) also converges to zero. Therefore, we have proved

$$\|A^{\frac{1}{2}}u_n\| = o(1), \quad (3.54)$$

which contradicts (3.10e).

Case 4. Let $(\alpha, \beta) \in R_4 \cup R_5 \cup S_I$, i.e.,

$$0 \leq \beta < \alpha, \quad \frac{1}{2} \leq \alpha, \quad \mu = \frac{\beta}{\alpha}.$$

By interpolation and (3.10g),

$$|\lambda_n|^{-\frac{\beta}{\alpha}} \|A^\beta(v_n + A^{\beta-\alpha}w_n)\| \leq \|\lambda_n^{-1}A^\alpha(v_n + A^{\beta-\alpha}w_n)\|^\frac{\beta}{\alpha} \|v_n + A^{\beta-\alpha}w_n\|^{1-\frac{\beta}{\alpha}} = O(1). \quad (3.55)$$

Applying bounded operator $A^{\beta-\alpha}$ to (3.9c) leads to

$$i\lambda_n|\lambda_n|^{-\frac{\beta}{\alpha}}A^{\beta-\alpha}w_n + |\lambda_n|^{-\frac{\beta}{\alpha}}A^\beta(v_n + A^{\beta-\alpha}w_n) = o(1). \quad (3.56)$$

It follows from (3.55)–(3.56) that

$$|\lambda_n|^\frac{\alpha-\beta}{\alpha} \|A^{\beta-\alpha}w_n\| = O(1). \quad (3.57)$$

Consequently,

$$\begin{aligned} \|w_n\| &= \|A^{\alpha-\beta}(A^{\beta-\alpha}w_n)\| \leq \|A^{\alpha-\frac{\beta}{2}}(A^{\beta-\alpha}w_n)\|^\frac{\alpha-\beta}{\alpha-\beta/2} \|A^{\beta-\alpha}w_n\|^\frac{\beta/2}{\alpha-\beta/2} \\ &= \|A^\frac{\beta}{2}w_n\|^\frac{2(\alpha-\beta)}{2\alpha-\beta} \|A^{\beta-\alpha}w_n\|^\frac{\beta}{2\alpha-\beta} \\ &= (|\lambda_n|^{-\frac{\beta}{2\alpha}} \|A^\frac{\beta}{2}w_n\|)^\frac{2(\alpha-\beta)}{2\alpha-\beta} (\lambda_n^\frac{\alpha-\beta}{\alpha} \|A^{\beta-\alpha}w_n\|)^\frac{\beta}{2\alpha-\beta} = o(1). \end{aligned} \quad (3.58)$$

Here, we have used (3.10d) and (3.57), and the identity

$$-\frac{\beta}{2\alpha} \frac{2(\alpha-\beta)}{2\alpha-\beta} + \frac{\alpha-\beta}{\alpha} \frac{\beta}{2\alpha-\beta} = 0.$$

Next, note that in region $R_4 \cup R_5 \cup S_I$, $1-\alpha < \frac{1}{2}$ and $1-2\alpha+\beta < 0$. By applying $|\lambda_n|^{\mu-1}A^{-\frac{1}{2}}$ to (3.9a), we see that

$$\|u_n\| = |\lambda_n|^{-1} \|v_n\| + o(1) = o(1).$$

Thus, by the boundedness of $\|A^\frac{1}{2}u_n\|$, making use of interpolation, one gets that $\|A^{1-\alpha}u_n\| = o(1)$. Moreover, we also have $\|A^{1-2\alpha+\beta}u_n\| = o(1)$.

We take the inner product of (3.9b) with $\lambda_n^{-1}|\lambda_n|^\frac{\beta}{\alpha}A^{1-2\alpha+\beta}u_n$ and (3.9c) with $\lambda_n^{-1}|\lambda_n|^\frac{\beta}{\alpha}A^{1-\alpha}u_n$ in H , respectively, to get the following:

$$i \langle v_n, A^{1-2\alpha+\beta}u_n \rangle + \|\lambda_n^{-1}A^{1-\alpha+\frac{\beta}{2}}u_n\|^2 - \lambda_n^{-1} \langle A^\beta w_n, A^{1-\alpha}u_n \rangle = o(1), \quad (3.59)$$

and

$$i \langle w_n, A^{1-\alpha}u_n \rangle + \lambda_n^{-1} \langle A^\frac{1}{2}v_n, A^\frac{1}{2}u_n \rangle + \lambda_n^{-1} \langle A^\beta w_n, A^{1-\alpha}u_n \rangle = o(1). \quad (3.60)$$

The first terms in (3.59) and (3.60) converge to zero, respectively. We can replace $\lambda_n^{-1}A^\frac{1}{2}v_n$ in (3.60) by $iA^\frac{1}{2}u_n$ due to (3.9a). Consequently, the sum of (3.59) and (3.60) yields

$$i\|A^\frac{1}{2}u_n\|^2 + \|\lambda_n^{-1}A^{1-\alpha+\frac{\beta}{2}}u_n\|^2 = o(1),$$

which implies

$$\|A^\frac{1}{2}u_n\| = o(1), \quad (3.61)$$

a contradiction to (3.10e) again. \square

Remark 3.3. In the region R_2 , $\mu = 2(2\alpha - \beta)$ stays unchanged on the line parallel to the common boundary of R_2 and R_1 , i.e., the line $\beta = 2\alpha - \frac{1}{2}$. It tends to 1 as the points in R_2 get closer to this common boundary. In the region R_3 , the situation is different since the common boundary of R_3 and R_1 is a single point. In this case, $\mu = 2(\beta + 2\alpha) - 2$ stays unchanged on the line parallel

to a part of the boundary of R_3 , i.e., $\beta = -2\alpha + 1$. It tends to 1 as the points in R_3 get closer to the common boundary of R_3 and R_1 . The most interesting case is the region $R_4 \cup R_5$ where $\mu = \frac{\beta}{\alpha}$ varies on the line parallel to the common boundary of R_4 and R_1 but stays unchanged on the lines passing the origin. It still tends to 1 as points in R_4 gets closer to the common boundary of R_4 and R_1 . Moreover, μ is continuous on the region $R_1 \cup R_2 \cup R_3 \cup R_4$. These observations make us to believe that the orders of *Gevrey* class obtained above are quite reasonable.

Remark 3.4. The smoothing region given in [15] does not include the region $R_5 = \{(\alpha, \beta) \mid 0 < \beta \leq 2\alpha - 1\}$. From the stability analysis in [11], system (1.2) is unstable in this region. However, the instability is caused by the fact that the origin becomes a spectral point of $\mathcal{A}_{\alpha, \beta}$, while the *Gevrey* class property relies on the behavior of spectrum and resolvent operator of $\mathcal{A}_{\alpha, \beta}$ near infinity.

4 Asymptotic Behavior of Eigenvalues

In this section, we are going to study the asymptotic behavior of some eigenvalue sequence for the operator $\mathcal{A}_{\alpha, \beta}$. This will lead to a proof of Theorem 2.4. Recall that we assume that there exists a sequence μ_n of eigenvalues of A such that

$$0 < \mu_1 \leq \mu_2 \leq \cdots, \quad \lim_{n \rightarrow \infty} \mu_n = \infty.$$

We now present the following lemma.

Lemma 4.1. *Let*

$$f(\lambda, \mu) = \lambda^3 + \lambda^2 \mu^\beta + \lambda(\mu + \mu^{2\alpha}) + \mu^{\beta+1}, \quad \forall (\lambda, \mu) \in \mathbb{C} \times \mathbb{R}_+. \quad (4.1)$$

If the following holds:

$$f(\lambda_n, \mu_n) = 0, \quad (4.2)$$

then λ_n is an eigenvalue of $\mathcal{A}_{\alpha, \beta}$.

Proof. For any $\lambda \in \mathbb{C}$, we consider the following equation for some non-zero $U = (u, v, w)^T \in \mathcal{D}(\mathcal{A}_{\alpha, \beta})$ such that

$$(\lambda - A_{\alpha, \beta})U = \begin{pmatrix} \lambda & -I & 0 \\ A & \lambda & -A^\alpha \\ 0 & A^\alpha & \lambda + A^\beta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \lambda u - v \\ Au + \lambda v - A^\alpha w \\ A^\alpha v + (\lambda + A^\beta)w \end{pmatrix} = 0. \quad (4.3)$$

Thus,

$$v = \lambda u,$$

$$w = A^{-\alpha}(Au + \lambda v) = A^{1-\alpha}u + \lambda A^{-\alpha}(\lambda u) = (A^{1-\alpha} + \lambda^2 A^{-\alpha})u,$$

and

$$\begin{aligned} 0 &= A^\alpha(\lambda u) + (\lambda + A^\beta)(A^{1-\alpha} + \lambda^2 A^{-\alpha})u \\ &= (\lambda A^\alpha + \lambda A^{1-\alpha} + A^{1+\beta-\alpha} + \lambda^3 A^{-\alpha} + \lambda^2 A^{\beta-\alpha})u \\ &= [\lambda^3 + \lambda^2 A^\beta + \lambda(A + A^{2\alpha}) + A^{\beta+1}]A^{-\alpha}u \equiv f(\lambda, A)A^{-\alpha}u, \end{aligned}$$

with $f(\cdot, \cdot)$ given by (4.1). Hence, if we take $u = \varphi_n$ to be an eigenvector of A corresponding to $\mu_n \in \sigma(A)$, and let

$$U_n(\lambda) = \begin{pmatrix} \varphi_n \\ \lambda \varphi_n \\ (\mu_n^{1-\alpha} + \lambda^2 \mu_n^{-\alpha}) \varphi_n \end{pmatrix},$$

then

$$(\lambda - \mathcal{A}_{\alpha,\beta})U_n(\lambda) = \begin{pmatrix} 0 \\ 0 \\ \mu_n^{-\alpha} f(\lambda, \mu_n) \varphi_n \end{pmatrix}.$$

Therefore, if λ_n is a root of $f(\lambda, \mu_n) = 0$, then λ_n is an eigenvalue of $\mathcal{A}_{\alpha,\beta}$. \square

Now, for any $n \geq 1$, we consider the following equation:

$$f(\lambda, \mu_n) \equiv \lambda^3 + \mu_n^\beta \lambda^2 + (\mu_n^{2\alpha} + \mu_n) \lambda + \mu_n^{\beta+1} = 0. \quad (4.4)$$

Let us denote

$$b_n = \mu_n^\beta, \quad c_n = \mu_n^{2\alpha} + \mu_n, \quad d_n = \mu_n^{\beta+1}. \quad (4.5)$$

Then (4.4) takes the following form:

$$\lambda^3 + b_n \lambda^2 + c_n \lambda + d_n = 0, \quad (4.6)$$

with $b_n, c_n, d_n \in \mathbb{R}_+$. Let

$$p_n = 3^2 c_n - 3 b_n^2, \quad q_n = 2 b_n^3 - 3^2 b_n c_n + 3^3 d_n. \quad (4.7)$$

Define

$$\begin{aligned} \Delta_n &= \left(\frac{q_n}{2}\right)^2 + \left(\frac{p_n}{3}\right)^3 = \left(b_n^3 - \frac{3^2}{2} b_n c_n + \frac{3^3}{2} d_n\right)^2 + (3 c_n - b_n^2)^3 \\ &= b_n^6 + \frac{3^4}{2^2} b_n^2 c_n^2 + \frac{3^6}{2^2} d_n^2 - 3^2 b_n^4 c_n + 3^3 b_n^3 d_n - \frac{3^5}{2} b_n c_n d_n + 3^3 c_n^3 - 3^3 c_n^2 b_n^2 + 3^2 c_n b_n^4 - b_n^6 \\ &= \frac{3^6}{2^2} d_n^2 + 3^3 b_n^3 d_n + 3^3 c_n^3 - \frac{3^5}{2} b_n c_n d_n - \frac{3^3}{2^2} b_n^2 c_n^2 \\ &= \frac{3^3}{2^2} (3^3 d_n^2 + 2^2 b_n^3 d_n + 2^2 c_n^3 - 2 \cdot 3^2 b_n c_n d_n - b_n^2 c_n^2) \\ &= \frac{27}{4} (27 d_n^2 + 4 b_n^3 d_n + 4 c_n^3 - 18 b_n c_n d_n - b_n^2 c_n^2), \end{aligned} \quad (4.8)$$

and

$$\Phi_{n,\pm} = -\frac{q_n}{2} \pm \sqrt{\Delta_n} \equiv -\frac{q_n}{2} \pm \sqrt{\left(\frac{q_n}{2}\right)^2 + \left(\frac{p_n}{3}\right)^3}. \quad (4.9)$$

With the above notations, we have the following result ([14]).

Proposition 4.2. (Cardano's Formula). Equation (4.6) admits three roots which are given by the following:

$$\lambda_k = \frac{1}{3} \left(\Phi_{n,+}^{\frac{1}{3}} \omega^k + \Phi_{n,-}^{\frac{1}{3}} \bar{\omega}^k - b_n \right), \quad k = 0, 1, 2, \quad (4.10)$$

with $\omega = e^{i\frac{2\pi}{3}} \equiv -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and for any $\zeta = |\zeta|e^{i\theta}$, we define $\zeta^{\frac{1}{3}} = |\zeta|^{\frac{1}{3}}e^{i\frac{\theta}{3}}$.

We note that in the case $\Delta_n > 0$, $\Phi_{n,\pm}$ are real. Consequently, the cubic equation (4.6) admits a unique real root, denoted by $\lambda_{n,0}$ and a pair of complex roots, denoted by $\lambda_{n,\pm}$. More precisely, in this case,

$$\begin{cases} \lambda_{n,0} = \frac{\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} - \mu_n^\beta}{3}, \\ \lambda_{n,\pm} = -\frac{\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + \mu_n^\beta}{6} \pm i \frac{\sqrt{3}(\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}})}{6}. \end{cases} \quad (4.11)$$

By the definition of b_n, c_n, d_n , we have

$$\begin{aligned}
\Delta_n &\equiv \Delta_n(\alpha, \beta) = \frac{27}{4} (27d_n^2 + 4b_n^3d_n + 4c_n^3 - 18b_nc_nd_n - b_n^2c_n^2) \\
&= \frac{27}{4} \left[27\mu_n^{2\beta+2} + 4\mu_n^{4\beta+1} + 4(\mu_n^{2\alpha} + \mu_n)^3 - 18\mu_n^{2\beta+1}(\mu_n^{2\alpha} + \mu_n) - \mu_n^{2\beta}(\mu_n^{2\alpha} + \mu_n)^2 \right] \\
&= \frac{27}{4} \left(27\mu_n^{2\beta+2} + 4\mu_n^{4\beta+1} + 4\mu_n^{6\alpha} + 12\mu_n^{4\alpha+1} + 12\mu_n^{2\alpha+2} + 4\mu_n^3 \right. \\
&\quad \left. - 18\mu_n^{2\alpha+2\beta+1} - 18\mu_n^{2\beta+2} - \mu_n^{4\alpha+2\beta} - 2\mu_n^{2\alpha+2\beta+1} - \mu_n^{2\beta+2} \right) \\
&= \frac{27}{4} \left(8\mu_n^{2\beta+2} + 4\mu_n^{4\beta+1} + 4\mu_n^{6\alpha} + 12\mu_n^{4\alpha+1} + 12\mu_n^{2\alpha+2} + 4\mu_n^3 - 20\mu_n^{2\alpha+2\beta+1} - \mu_n^{4\alpha+2\beta} \right) \\
&= 54\mu_n^{2\beta+2} + 27\mu_n^{4\beta+1} + 27\mu_n^{6\alpha} + 81\mu_n^{4\alpha+1} + 81\mu_n^{2\alpha+2} + 27\mu_n^3 - 135\mu_n^{2\alpha+2\beta+1} - \frac{27}{4}\mu_n^{4\alpha+2\beta},
\end{aligned}$$

and

$$\begin{aligned}
q_n &\equiv q_n(\alpha, \beta) = 2b_n^3 - 3^2b_nc_n + 3^3d_n = 2\mu_n^{3\beta} - 9\mu_n^\beta(\mu_n^{2\alpha} + \mu_n) + 27\mu_n^{\beta+1} \\
&= 2\mu_n^{3\beta} - 9\mu_n^{2\alpha+\beta} + 18\mu_n^{\beta+1}.
\end{aligned}$$

Our first result is about the leading term in $\Delta_n(\alpha, \beta)$ and in $q_n(\alpha, \beta)$.

Lemma 4.3. *The following hold:*

$$\Delta_n(\alpha, \beta) = \begin{cases} 27\mu_n^{4\beta+1}(1 + o(1)), & (\alpha, \beta) \in R_2 \cup S_1, \\ 216\mu_n^3(1 + o(1)), & (\alpha, \beta) \in R_3, \quad \alpha = \frac{1}{2}, \quad 0 \leq \beta < \frac{1}{2}, \\ 108\mu_n^3(1 + o(1)), & (\alpha, \beta) \in R_3 \cup S_2, \quad 0 \leq \alpha < \frac{1}{2}, \quad \beta = \frac{1}{2}, \\ 27\mu_n^3(1 + o(1)), & (\alpha, \beta) \in R_3 \cup S_2, \quad 0 \leq \alpha, \beta < \frac{1}{2}, \\ 27\mu_n^{6\alpha}(1 + o(1)), & (\alpha, \beta) \in R_4 \cup R_5 \cup S_I, \end{cases} \quad (4.12)$$

and

$$q_n(\alpha, \beta) = \begin{cases} 2\mu_n^{3\beta}(1 + o(1)), & (\alpha, \beta) \in R_2 \cup S_1, \\ 9\mu_n^{\beta+1}(1 + o(1)), & (\alpha, \beta) \in R_3, \quad \alpha = \frac{1}{2}, \quad 0 \leq \beta < \frac{1}{2}, \\ 20\mu_n^{\frac{3}{2}}(1 + o(1)), & (\alpha, \beta) \in R_3 \cup S_2, \quad 0 \leq \alpha < \frac{1}{2}, \quad \beta = \frac{1}{2}, \\ 18\mu_n^{\beta+1}(1 + o(1)), & (\alpha, \beta) \in R_3 \cup S_2, \quad 0 \leq \alpha, \beta < \frac{1}{2}, \\ -9\mu_n^{2\alpha+\beta}(1 + o(1)), & (\alpha, \beta) \in R_4 \cup R_5 \cup S_I. \end{cases} \quad (4.13)$$

Proof. For $(\alpha, \beta) \in R_2, \frac{1}{2} \leq \alpha < \beta \leq 1$, we have

$$4\beta + 1 > 4\alpha + 2\beta \begin{cases} \geq 2\alpha + 2\beta + 1 \geq 2\beta + 2, \\ > 6\alpha \geq 4\alpha + 1 \geq 2\alpha + 2 \geq 3. \end{cases}$$

For $(\alpha, \beta) \in R_2 \cup S_1, 0 \leq \alpha < \frac{1}{2} < \beta$,

$$4\beta + 1 > 2\beta + 2 > \begin{cases} 2\alpha + 2\beta + 1 > 4\alpha + 2\beta, \\ 3 > 2\alpha + 2 > 4\alpha + 1 > 6\alpha. \end{cases}$$

Hence, in $R_2 \cup S_1$,

$$\Delta_n(\alpha, \beta) = 27\mu_n^{4\beta+1}(1 + o(1)).$$

Also, for $(\alpha, \beta) \in R_2 \cup S_1$, $\beta > \frac{1}{2} \vee \alpha$. Thus,

$$3\beta > (2\alpha + \beta) \vee (\beta + 1),$$

which implies that

$$q_n(\alpha, \beta) = 2\mu_n^{3\beta}(1 + o(1)).$$

For $(\alpha, \beta) \in R_3 \cup S_2$, $0 \leq \alpha, \beta \leq \frac{1}{2}$, $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$. We look at three different cases.

For $(\alpha, \beta) \in R_3$ with $\alpha = \frac{1}{2}$, and $\beta < \frac{1}{2}$, we have

$$\Delta_n\left(\frac{1}{2}, \beta\right) = \frac{27}{4}\left(32\mu_n^3 + 4\mu_n^{4\beta+1} - 13\mu_n^{2\beta+2}\right),$$

whose leading term is $216\mu_n^3$ since

$$3 > 2\beta + 2 > 4\beta + 1.$$

Also,

$$q_n\left(\frac{1}{2}, \beta\right) = 2\mu_n^{3\beta} + 9\mu_n^{\beta+1},$$

whose leading term is $9\mu_n^{\beta+1}$ since $\beta < \frac{1}{2}$.

For $(\alpha, \beta) \in R_3 \cup S_2$ with $\beta = \frac{1}{2}$, $0 \leq \alpha < \frac{1}{2}$, we have

$$\Delta_n\left(\alpha, \frac{1}{2}\right) = \frac{27}{4}\left(16\mu_n^3 + 4\mu_n^{6\alpha} + 11\mu_n^{4\alpha+1} - 8\mu_n^{2\alpha+2}\right),$$

whose leading term is $108\mu_n^3$ since

$$3 > 2\alpha + 2 > 4\alpha + 1 > 6\alpha.$$

Also,

$$q_n\left(\alpha, \frac{1}{2}\right) = 20\mu_n^{\frac{3}{2}} - 9\mu_n^{2\alpha+\frac{1}{2}},$$

whose leading term is $20\mu_n^{\frac{3}{2}}$ since $\alpha < \frac{1}{2}$.

Now for $(\alpha, \beta) \in R_3 \cup S_2$ with $\alpha, \beta < \frac{1}{2}$, we have

$$3 > \begin{cases} 2\alpha + 2 > \begin{cases} 4\alpha + 1 > 6\alpha, \\ 2\alpha + 2\beta + 1 > 4\alpha + 2\beta, \end{cases} \\ 4\beta + 1 > 2\beta + 2. \end{cases}$$

Hence, the leading term of $\Delta_n(\alpha, \beta)$ is $27\mu_n^3$. Also, since

$$\beta + 1 > (3\beta) \vee (2\alpha + \beta),$$

the leading term in $q_n(\alpha, \beta)$ is $18\mu_n^{\beta+1}$.

Finally, in $R_4 \cup R_5 \cup S_I$, $0 \leq \beta \vee \frac{1}{2} < \alpha \leq 1$, we have

$$6\alpha > \begin{cases} 4\alpha + 1 > 2\alpha + 2 > 3, \\ 4\alpha + 2\beta > \begin{cases} 2\alpha + 2\beta + 1 > 2\beta + 2, \\ 2\alpha + 4\beta > 4\beta + 1. \end{cases} \end{cases}$$

Thus, the leading term of $\Delta_n(\alpha, \beta)$ is $27\mu_n^{6\alpha}$. Also, since

$$2\alpha + \beta > (3\beta) \vee (\beta + 1),$$

the leading term of $q_n(\alpha, \beta)$ is $-9\mu_n^{2\alpha+\beta}$. □

The following gives the asymptotic behavior of the solutions to (4.4).

Theorem 4.4. *Let the assumption of Theorem 2.4 hold. Let $n \geq 1$ be large enough. Then*

$$\Delta_n(\alpha, \beta) > 0, \quad \forall (\alpha, \beta) \in R_2 \cup R_3 \cup R_4 \cup R_5 \cup S_1 \cup S_2 \cup S_I, \quad (4.14)$$

and (4.4) admits a real root $\lambda_{n,0}$ and a pair of conjugate complex roots $\lambda_{n,\pm}$. Moreover, the following asymptotic behavior will hold:

(i) For $(\alpha, \beta) \in R_2 \cup S_1$,

$$\begin{cases} \lambda_{n,0} = -\mu_n^\beta(1 + o(1)), \\ \lambda_{n,\pm} = -\frac{1}{2}\mu_n^{2\alpha-\beta}(1 + o(1)) \pm i\mu_n^{\frac{1}{2}}(1 + o(1)). \end{cases} \quad (4.15)$$

(ii) For $(\alpha, \beta) \in R_3 \cup S_2$,

$$\begin{cases} \lambda_{n,0} = \begin{cases} -\mu_n^\beta(1 + o(1)), & \alpha < \frac{1}{2}, \\ -\frac{1}{2}\mu_n^\beta(1 + o(1)), & \alpha = \frac{1}{2}, \end{cases} \\ \lambda_{n,\pm} = \begin{cases} -\frac{1}{2}\mu_n^{2\alpha+\beta-1}(1 + o(1)) \pm i\mu_n^{\frac{1}{2}}(1 + o(1)), & \alpha, \beta < \frac{1}{2}, \\ -\frac{1}{4}\mu_n^\beta(1 + o(1)) \pm i\sqrt{2}\mu_n^{\frac{1}{2}}(1 + o(1)), & \alpha = \frac{1}{2}, \\ -\frac{1}{4}\mu_n^{2\alpha-\frac{1}{2}}(1 + o(1)) \pm \mu_n^{\frac{1}{2}}(1 + o(1)), & \beta = \frac{1}{2}. \end{cases} \end{cases} \quad (4.16)$$

(iii) In region $R_4 \cup R_5 \cup S_I$,

$$\begin{cases} \lambda_{n,0} = -\mu_n^{1+\beta-2\alpha}(1 + o(1)), \\ \lambda_{n,\pm} = -\frac{1}{2}\mu_n^\beta(1 + o(1)) \pm i\mu_n^\alpha(1 + o(1)). \end{cases} \quad (4.17)$$

Proof. By Lemma 4.3, we have (4.14). Therefore, the cubic equation (4.4) has one real root and a pair of complex conjugate roots when n is large enough:

$$\begin{cases} \lambda_{n,0} = \frac{1}{3}(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} - \mu_n^\beta), \\ \lambda_{n,\pm} = -\frac{1}{6}(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^\beta) \pm i\frac{\sqrt{3}}{6}(\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}}). \end{cases} \quad (4.18)$$

In what follows, we are going to find the leading terms of the real and imaginary part of the root expression in (4.18).

Case 1: $(\alpha, \beta) \in R_2 \cup S_1$, In this case, one has

$$\Delta_n = 27\mu_n^{4\beta+1}(1+o(1)), \quad q_n = 2\mu_n^{3\beta}(1+\omega(1)).$$

Thus,

$$\sqrt{\Delta_n} = 3\sqrt{3}\mu_n^{2\beta+\frac{1}{2}}(1+o(1)).$$

Then

$$\Phi_{n,\pm} = -\frac{q_n}{2} \pm \sqrt{\Delta_n} = -\mu_n^{3\beta}(1+o(1)) \pm 3\sqrt{3}\mu_n^{2\beta+\frac{1}{2}}(1+o(1)) = -\mu_n^{3\beta}(1+o(1)),$$

since for $(\alpha, \beta) \in R_2 \cup S_1$,

$$3\beta > 2\beta + \frac{1}{2}.$$

Therefore,

$$\begin{aligned} \Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}} &= \frac{\Phi_{n,+} - \Phi_{n,-}}{\Phi_{n,+}^{\frac{2}{3}} + \Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{2}{3}}} = \frac{2\sqrt{\Delta_n}}{\Phi_{n,+}^{\frac{2}{3}} + \Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{2}{3}}} \\ &= \frac{6\sqrt{3}\mu_n^{2\beta+\frac{1}{2}}(1+o(1))}{3\mu_n^{2\beta}(1+o(1))} = 2\sqrt{3}\mu_n^{\frac{1}{2}}(1+o(1)), \end{aligned} \quad (4.19)$$

and

$$\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} = -2\mu_n^{\beta}(1+o(1)). \quad (4.20)$$

Consequently,

$$\lambda_{n,0} = \frac{1}{3}(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} - \mu_n^{\beta}) = -\mu_n^{\beta}(1+o(1)).$$

Also,

$$\begin{aligned} \lambda_{n,\pm} &= -\frac{1}{6}(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^{\beta}) \pm i\frac{\sqrt{3}}{6}(\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}}) \\ &= -\frac{1}{6}(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^{\beta}) \pm i\mu_n^{\frac{1}{2}}(1+o(1)). \end{aligned}$$

Note that the real part of $\lambda_{n,\pm}$ above cannot be estimated using the above argument due to cancelation of the leading term μ_n^{β} . Therefore, we take a different approach. To this end, we denote

$$\Lambda_{n,0} = 2\operatorname{Re} \lambda_{n,\pm}.$$

By the Vieta's formula for the cubic equation (4.4), we have

$$-\mu_n^{\beta} = \lambda_{n,0} + \lambda_{n,+} + \lambda_{n,-} = \lambda_{n,0} + 2\operatorname{Re} \lambda_{n,\pm} = \lambda_{n,0} + \Lambda_{n,0}. \quad (4.21)$$

Therefore, $\lambda_{n,0} = -\Lambda_{n,0} - \mu_n^{\beta}$ satisfies (4.4), i.e.,

$$\begin{aligned} 0 &= \lambda_{n,0}^3 + \mu_n^{\beta}\lambda_{n,0}^2 + (\mu_n^{2\alpha} + \mu_n)\lambda_{n,0} + \mu_n^{\beta+1} \\ &= -(\Lambda_{n,0} + \mu_n^{\beta})^3 + \mu_n^{\beta}(\Lambda_{n,0} + \mu_n^{\beta})^2 - (\mu_n^{2\alpha} + \mu_n)(\Lambda_{n,0} + \mu_n^{\beta}) + \mu_n^{\beta+1} \\ &= -\Lambda_{n,0}^3 - 3\mu_n^{\beta}\Lambda_{n,0}^2 - 3\mu_n^{2\beta}\Lambda_{n,0} - \mu_n^{3\beta} + \mu_n^{\beta}\Lambda_{n,0}^2 + 2\mu_n^{2\beta}\Lambda_{n,0} + \mu_n^{3\beta} \\ &\quad - (\mu_n^{2\alpha} + \mu_n)\Lambda_{n,0} - \mu_n^{2\alpha+\beta} - \mu_n^{\beta+1} + \mu_n^{\beta+1} \\ &= -\Lambda_{n,0}^3 - 2\mu_n^{\beta}\Lambda_{n,0}^2 - (\mu_n^{2\alpha} + \mu_n^{2\beta} + \mu_n)\Lambda_{n,0} - \mu_n^{2\alpha+\beta}. \end{aligned}$$

This means that $\Lambda_{n,0}$ is a real solution to the following new cubic equation

$$\Lambda^3 + 2\mu_n^\beta \Lambda^2 + (\mu_n^{2\alpha} + \mu_n^{2\beta} + \mu_n) \Lambda + \mu_n^{2\alpha+\beta} = 0. \quad (4.22)$$

Next, by defining

$$\Lambda_{n,\pm} = -\mu_n^\beta - \lambda_{n,\pm},$$

and by the fact that $\lambda_{n,\pm}$ are the roots of (4.4), using the same argument as above, we see that $\Lambda_{n,\pm}$ is a pair of conjugate complex roots of (4.22). Now, we rewrite (4.21) as follows:

$$\Lambda_{n,0} = -\lambda_{n,0} - \mu_n^\beta = 2\operatorname{Re}\lambda_{n,\pm}.$$

Since the leading term of $\operatorname{Re}\lambda_{n,\pm}$ is $o(\mu_n^\beta)$, the complex roots of equation (4.22) satisfy

$$\Lambda_{n,\pm} = -\mu_n^\beta - \lambda_{n,\pm} = -\mu_n^\beta(1 + o(1)) \mp i\mu_n^{\frac{1}{2}}(1 + o(1)).$$

Then by the Vieta's formula for equation (4.22), one has (noting $\beta > \frac{1}{2}$)

$$-\mu_n^{2\alpha+\beta} = \Lambda_{n,0}\Lambda_{n,+}\Lambda_{n,-} = \Lambda_{n,0}\mu_n^{2\beta}(1 + o(1)).$$

Therefore,

$$\operatorname{Re}\lambda_{n,\pm} = \frac{1}{2}\Lambda_{n,0} = -\frac{1}{2}\mu_n^{2\alpha-\beta}(1 + o(1)).$$

Case 2: $(\alpha, \beta) \in R_4 \cup R_5 \cup S_I$. In this case,

$$\Delta_n = 27\mu_n^{6\alpha}(1 + o(1)), \quad q_n = -9\mu_n^{2\alpha+\beta}(1 + o(1)).$$

Then,

$$\sqrt{\Delta_n} = 3\sqrt{3}\mu_n^{3\alpha}(1 + o(1)).$$

This leads to

$$\Phi_{n,\pm} = -\frac{q_n}{2} \pm \sqrt{\Delta_n} = \frac{9}{2}\mu_n^{2\alpha+\beta}(1 + o(1)) \pm 3\sqrt{3}\mu_n^{3\alpha}(1 + o(1)) = \pm 3\sqrt{3}\mu_n^{3\alpha}(1 + o(1)),$$

since for the current case, $\beta < \alpha$ which implies

$$3\alpha > 2\alpha + \beta.$$

Hence,

$$\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}} = 2\Phi_{n,+}^{\frac{1}{3}} = 2\sqrt{3}\mu_n^\alpha(1 + o(1)), \quad (4.23)$$

and

$$\begin{aligned} \Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} &= \frac{\Phi_{n,+} + \Phi_{n,-}}{\Phi_{n,+}^{\frac{2}{3}} - \Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{2}{3}}} = \frac{-q_n}{\Phi_{n,+}^{\frac{2}{3}} - \Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{2}{3}}} \\ &= \frac{9\mu_n^{2\alpha+\beta}(1 + o(1))}{9\mu_n^{2\alpha}(1 + o(1))} = \mu_n^\beta(1 + o(1)). \end{aligned} \quad (4.24)$$

Consequently,

$$\lambda_{n,\pm} = -\frac{1}{6}\left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^\beta\right) \pm i\frac{\sqrt{3}}{6}\left(\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}}\right) = -\frac{1}{2}\mu_n^\beta(1 + o(1)) \pm i\mu_n^\alpha(1 + o(1)).$$

For the real root $\lambda_{n,0}$, we will have cancelation of the leading term μ_n^β . Therefore, we may let

$$\lambda_{n,0} = c\mu_n^\xi(1 + o(1)),$$

for some $c \in \mathbb{R}$ and $0 < \xi < \beta$. Then by Vieta's formula, noting $\beta < \alpha$,

$$-\mu_n^{\beta+1} = \lambda_{n,0}\lambda_{n,+}\lambda_{n,-} = c\mu_n^\xi\left(\frac{1}{4}\mu_n^{2\beta} + \mu_n^{2\alpha}\right)(1 + o(1)) = c\mu_n^{2\alpha+\xi}(1 + o(1)).$$

Consequently, it is necessary that

$$c = -1, \quad \xi = \beta + 1 - 2\alpha.$$

Case 3: $(\alpha, \beta) \in R_3 \cup S_2$. We will consider three subcases.

Subcase 1: $(\alpha, \beta) \in R_3 \cup S_2$, $0 \leq \alpha, \beta < \frac{1}{2}$. In this case,

$$\Delta_n = 27\mu_n^3(1 + o(1)), \quad q_n = 18\mu_n^{\beta+1}(1 + o(1)\beta).$$

Then,

$$\sqrt{\Delta_n} = 3\sqrt{3}\mu_n^{\frac{3}{2}}(1 + o(1)).$$

This further gives

$$\Phi_{n,\pm} = -\frac{q_n}{2} \pm \sqrt{\Delta_n} = -9\mu_n^{\beta+1}(1 + o(1)) \pm 3\sqrt{3}\mu_n^{\frac{3}{2}}(1 + o(1)) = \pm 3\sqrt{3}\mu_n^{\frac{3}{2}}(1 + o(1)),$$

since for the current case,

$$\frac{3}{2} > \beta + 1.$$

Then

$$\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}} = 2\sqrt{3}\mu_n^{\frac{1}{2}}(1 + o(1)), \quad (4.25)$$

and

$$\begin{aligned} \Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} &= \frac{\Phi_{n,+} + \Phi_{n,-}}{\Phi_{n,+}^{\frac{2}{3}} - \Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{2}{3}}} = \frac{-q_n}{\Phi_{n,+}^{\frac{2}{3}} - \Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{2}{3}}} \\ &= \frac{-18\mu_n^{\beta+1}(1 + o(1))}{9\mu_n(1 + o(1))} = -2\mu_n^\beta(1 + o(1)). \end{aligned} \quad (4.26)$$

Consequently,

$$\lambda_{n,0} = \frac{1}{3}\left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} - \mu_n^\beta\right) = -\mu_n^\beta(1 + o(1)),$$

and

$$\begin{aligned} \lambda_{n,\pm} &= -\frac{1}{6}\left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^\beta\right) \pm i\frac{\sqrt{3}}{6}\left(\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}}\right) \\ &= -\frac{1}{6}\left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^\beta\right) \pm i\mu_n^{\frac{1}{2}}(1 + o(1)). \end{aligned}$$

From the above, we see that the leading terms in $\text{Re } \lambda_{n,\pm}$ are canceled. Thus,

$$\Lambda_{n,0} \equiv 2\text{Re } \lambda_{n,\pm} = o(\mu_n^\beta).$$

The same as Case 1, $\Lambda_{n,0}$ is a real root of cubic equation (4.22), and

$$\Lambda_{n,\pm} = -\mu_n^\beta - \lambda_{n,\pm} = -\mu_n^\beta - \text{Re } \lambda_{n,\pm} \pm i\mu_n^{\frac{1}{2}}(1 + o(1)) = -\mu_n^\beta(1 + o(1)) \pm i\mu_n^{\frac{1}{2}}(1 + o(1))$$

are the pair of conjugate complex roots of (4.22). Further, by Vieta's formula for the equation (4.22), one obtains

$$\begin{aligned} -\mu_n^{2\alpha+\beta} &= \Lambda_{n,0}\Lambda_{n,+}\Lambda_{n,-} = \Lambda_{n,0}\left[\left(\operatorname{Re} \Lambda_{n,\pm}\right)^2 + \left(\operatorname{Im} \Lambda_{n,\pm}\right)^2\right] \\ &= \Lambda_{n,0}\left[\mu_n^{2\beta}(1+o(1)) + \mu_n(1+o(1))\right] = \Lambda_{n,0}\mu_n(1+o(1)), \end{aligned}$$

since $\beta < \frac{1}{2}$. Consequently,

$$\Lambda_{n,0} = -\mu_n^{2\alpha+\beta-1}(1+o(1)).$$

Hence,

$$\operatorname{Re} \lambda_{n,\pm} = \frac{1}{2}\Lambda_{n,0} = -\frac{1}{2}\mu_n^{2\alpha+\beta-1}(1+o(1)),$$

proving our claim.

Subcase 2: $(\alpha, \beta) \in R_3$, $\alpha = \frac{1}{2}$, $\beta < \frac{1}{2}$. For this case,

$$\Delta_n = 216\mu_n^3(1+o(1)), \quad q_n = 9\mu_n^{\beta+1}(1+o(1)).$$

Then

$$\Phi_{n,\pm} = -\frac{q_n}{2} \pm \sqrt{\Delta_n} = -\frac{9}{2}\mu_n^{\beta+1}(1+o(1)) \pm 6\sqrt{6}\mu_n^{\frac{3}{2}}(1+o(1)) = \pm 6\sqrt{6}\mu_n^{\frac{3}{2}}(1+o(1)),$$

since

$$\frac{3}{2} > \beta + 1.$$

We have

$$\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}} = 2\sqrt{6}\mu_n^{\frac{1}{2}}(1+o(1)), \quad (4.27)$$

and

$$\begin{aligned} \Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} &= \frac{\Phi_{n,+} + \Phi_{n,-}}{\Phi_{n,+}^{\frac{2}{3}} - \Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{2}{3}}} = \frac{-q_n}{\Phi_{n,+}^{\frac{2}{3}} - \Phi_{n,+}^{\frac{1}{3}}\Phi_{n,-}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{2}{3}}} \\ &= \frac{-9\mu_n^{\beta+1}(1+o(1))}{18\mu_n(1+o(1))} = -\frac{1}{2}\mu_n^{\beta}(1+o(1)). \end{aligned} \quad (4.28)$$

Consequently,

$$\lambda_{n,0} = \frac{1}{3}\left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} - \mu_n^{\beta}\right) = -\frac{1}{2}\mu_n^{\beta}(1+o(1)),$$

and

$$\begin{aligned} \lambda_{n,\pm} &= -\frac{1}{6}\left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^{\beta}\right) \pm i\frac{\sqrt{3}}{6}\left(\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}}\right) \\ &= -\frac{1}{4}\mu_n^{\beta} \pm i\sqrt{2}\mu_n^{\frac{1}{2}}(1+o(1)). \end{aligned}$$

Subcase 3: $(\alpha, \beta) \in R_3 \cup S_2$, $\beta = \frac{1}{2}$ and $\alpha < \frac{1}{2}$. In this case,

$$\Delta_n = 108\mu_n^3(1+o(1)), \quad q_n = 20\mu_n^{\frac{3}{2}}(1+o(1)).$$

Then

$$\Phi_{n,\pm} = -\frac{q_n}{2} \pm \sqrt{\Delta_n} = (-10 \pm 6\sqrt{3})\mu_n^{\frac{3}{2}}(1+o(1)) = (-1 \pm \sqrt{3})^3\mu_n^{\frac{3}{2}}(1+o(1)).$$

Hence,

$$\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}} = \left[(-1 + \sqrt{3}) - (-1 - \sqrt{3})\right] \mu_n^{\frac{1}{2}}(1 + o(1)) = 2\sqrt{3}\mu_n^{\frac{1}{2}}(1 + o(1)), \quad (4.29)$$

and

$$\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} = \left[(-1 + \sqrt{3}) + (-1 - \sqrt{3})\right] \mu_n^{\frac{1}{2}}(1 + o(1)) = -2\mu_n^{\frac{1}{2}}(1 + o(1)). \quad (4.30)$$

Consequently,

$$\lambda_{n,0} = \frac{1}{3} \left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} - \mu_n^{\frac{1}{2}} \right) = -\mu_n^{\frac{1}{2}}(1 + o(1)),$$

and

$$\begin{aligned} \lambda_{n,\pm} &= -\frac{1}{6} \left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^{\frac{1}{2}} \right) \pm i \frac{\sqrt{3}}{6} \left(\Phi_{n,+}^{\frac{1}{3}} - \Phi_{n,-}^{\frac{1}{3}} \right) \\ &= -\frac{1}{6} \left(\Phi_{n,+}^{\frac{1}{3}} + \Phi_{n,-}^{\frac{1}{3}} + 2\mu_n^{\frac{1}{2}} \right) \pm i \mu_n^{\frac{1}{2}}(1 + o(1)). \end{aligned}$$

Once again, we see that the leading terms in $\text{Re } \lambda_{n,\pm}$ are canceled out. Therefore,

$$\text{Re } \lambda_{n,\pm} = o(\mu_n^{\frac{1}{2}}).$$

Mimicking Case 1, we know that

$$\begin{cases} \Lambda_{n,0} = 2\text{Re } \lambda_{n,\pm}, \\ \Lambda_{n,\pm} = -\mu_n^{\frac{1}{2}} - \lambda_{n,\pm} = -\mu_n^{\frac{1}{2}}(1 + o(1)) \mp \mu_n^{\frac{1}{2}}(1 + o(1)) \end{cases}$$

are three roots of the cubic equation (4.22). Then by the Vieta's formula,

$$-\mu_n^{2\alpha+\frac{1}{2}} = \Lambda_{n,0}\Lambda_{n,+}\Lambda_{n,-} = \Lambda_{n,0}[2\mu_n + (1 + o(1))].$$

Therefore,

$$\Lambda_{n,0} = -\frac{1}{2}\mu_n^{2\alpha-\frac{1}{2}}(1 + o(1)),$$

i.e.,

$$\text{Re } \lambda_{n,\pm} = \frac{1}{2}\Lambda_{n,0} = -\frac{1}{4}\mu_n^{2\alpha-\frac{1}{2}}(1 + o(1)).$$

This completes the proof. \square

We see easily that (2.19)–(2.20) follows from Theorem 4.4. Therefore, proof of Theorem 2.4 follows immediately.

Remark 4.5. In our previous paper [11], a complete stability analysis for system (1.1) was presented. The asymptotic expressions of eigenvalues in $\lambda_{n,0}$ and $\lambda_{n,\pm}$ for $(\alpha, \beta) \in S_1 \cup S_2 \cup S_I$ were derived by plugging the Taylor series expansion of $\Phi_{n,\pm}^{\frac{1}{3}}$ into (4.11). Due to the cancelation of leading term and other terms, this method became cumbersome in finding an explicit ordering of the power terms of μ_n in each region. A number of subregions were further introduced, but the argument there were not clear and satisfactory. The idea used in the current paper is much better and it enable us to present a complete analysis of the asymptotic behavior of the eigenvalues.

We now present an interesting corollary of Theorem 4.4, which also gives us an impression that the index $\mu(\alpha, \beta)$ is sharp.

Corollary 4.6. *Under the assumption of Theorem 2.4, the following holds:*

$$\overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} |\lambda|^{\mu(\alpha, \beta)} \|(i\lambda - \mathcal{A}_{\alpha, \beta})^{-1}\| \geq 2, \quad \forall (\alpha, \beta) \in [0, 1] \times [0, 1] \setminus R_1. \quad (4.31)$$

Proof. First of all, we claim that if $\lambda \in \mathbb{C}$, with $\operatorname{Re} \lambda \neq 0$, is an eigenvalue of \mathcal{A} which is a densely defined closed operator on some Hilbert space \mathcal{H} such that $(i\operatorname{Im} \lambda - \mathcal{A})^{-1}$ exists. Then

$$|\operatorname{Re} \lambda| \|(i\operatorname{Im} \lambda - \mathcal{A})^{-1}\| \geq 1. \quad (4.32)$$

In fact, there exists an $x \in \mathcal{D}(\mathcal{A})$ with $\|x\| = 1$ such that

$$\mathcal{A}x = (\mu + i\nu)x.$$

Hence,

$$\mu(i\nu - \mathcal{A})^{-1}x = -x.$$

Thus (4.32) follows. Now, from Theorem 4.4, we know that $\mathcal{A}_{\alpha, \beta}$ has a sequence of conjugate complex eigenvalues of the following form:

$$\lambda_{n, \pm} = -a\mu_n^\xi(1 + o(1)) \pm ib\mu_n^\eta(1 + o(1)),$$

for some real constants $a, b, \eta > 0$ and $\xi \geq 0$. Let $\lambda = b\mu_n^\eta$. By (4.32), we have

$$1 \leq a\mu_n^\xi(1 + o(1)) \|(ib\mu_n^\eta - \mathcal{A}_{\alpha, \beta})^{-1}\| = \frac{a}{b^{\frac{\xi}{\eta}}} |\lambda|^{\frac{\xi}{\eta}} \|(i\lambda - \mathcal{A})^{-1}\| (1 + o(1)).$$

Hence,

$$|\lambda|^{\frac{\xi}{\eta}} \|(i\lambda - \mathcal{A}_{\alpha, \beta})^{-1}\| \geq \frac{b^{\xi/\eta}}{a} (1 + o(1)).$$

Now, we look at different regions.

In region $R_2 \cup S_1$,

$$\lambda_{n, \pm} = -\frac{1}{2}\mu_n^{2\alpha-\beta}(1 + o(1)) \pm i\mu_n^{\frac{1}{2}}(1 + o(1)).$$

Thus,

$$\frac{\xi}{\eta} = 2(2\alpha - \beta), \quad \frac{b^{\xi/\eta}}{a} = 2,$$

which leads to

$$|\lambda|^{2(2\alpha-\beta)} \|(i\lambda - \mathcal{A}_{\alpha, \beta})^{-1}\| \geq 2(1 + o(1)).$$

In region $R_3 \cup S_2$,

$$\lambda_{n, \pm} = \begin{cases} -\frac{1}{2}\mu_n^{2\alpha+\beta-1}(1 + o(1)) \pm i\mu_n^{\frac{1}{2}}(1 + o(1)), & \alpha, \beta < \frac{1}{2}, \\ -\frac{1}{4}\mu_n^\beta(1 + o(1)) \pm i\sqrt{2}\mu_n^{\frac{1}{2}}(1 + o(1)), & \alpha = \frac{1}{2}, \\ -\frac{1}{4}\mu_n^{2\alpha-\frac{1}{2}}(1 + o(1)) \pm \mu_n^{\frac{1}{2}}(1 + o(1)), & \beta = \frac{1}{2}. \end{cases} \quad (4.33)$$

Hence,

$$\begin{cases} \frac{\xi}{\eta} = 2(2\alpha + \beta - 1), & \frac{b^{\xi/\eta}}{a} = 2, & \alpha, \beta < \frac{1}{2}, \\ \frac{\xi}{\eta} = 2\beta, & \frac{b^{\xi/\eta}}{a} = 2^{\beta+2}, & \alpha = \frac{1}{2}, \\ \frac{\xi}{\eta} = 4\alpha - 1, & \frac{b^{\xi/\eta}}{a} = 4, & \beta = \frac{1}{2}. \end{cases}$$

Consequently,

$$|\lambda|^{2(2\alpha+\beta-1)} \|(i\lambda - \mathcal{A}_{\alpha,\beta})^{-1}\| \geq 2.$$

Finally, in $R_4 \cup R_5 \cup S_I$,

$$\lambda_{n,\pm} = -\frac{1}{2}\mu_n^\beta(1 + o(1)) \pm i\mu_n^\alpha(1 + o(1)).$$

Thus,

$$\frac{\xi}{\eta} = \frac{\beta}{\alpha}, \quad \frac{b^{\xi/\eta}}{a} = 2,$$

and we again have

$$|\lambda|^{\frac{\beta}{\alpha}} \|(i\lambda - \mathcal{A}_{\alpha,\beta})^{-1}\| \geq 2(1 + o(1)).$$

Combining the above, we see that for any given $(\alpha, \beta) \in [0, 1] \times [0, 1]$,

$$|\lambda|^{\mu(\alpha,\beta)} \|(i\lambda - \mathcal{A}_{\alpha,\beta})^{-1}\| \geq 2(1 + o(1)),$$

with $\mu(\alpha, \beta)$ given by (2.17). Hence, (4.31) follows. \square

To conclude this paper, we point out that with the complete stability and regularity results for system (1.1), we should be able to consider the more general system (1.7) when the operators B and D are equivalent (in a certain sense) to A^α and A^β , respectively. Such a general setting will allow differential operators to have different boundary conditions. Relevant results will be addressed in a forthcoming paper.

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